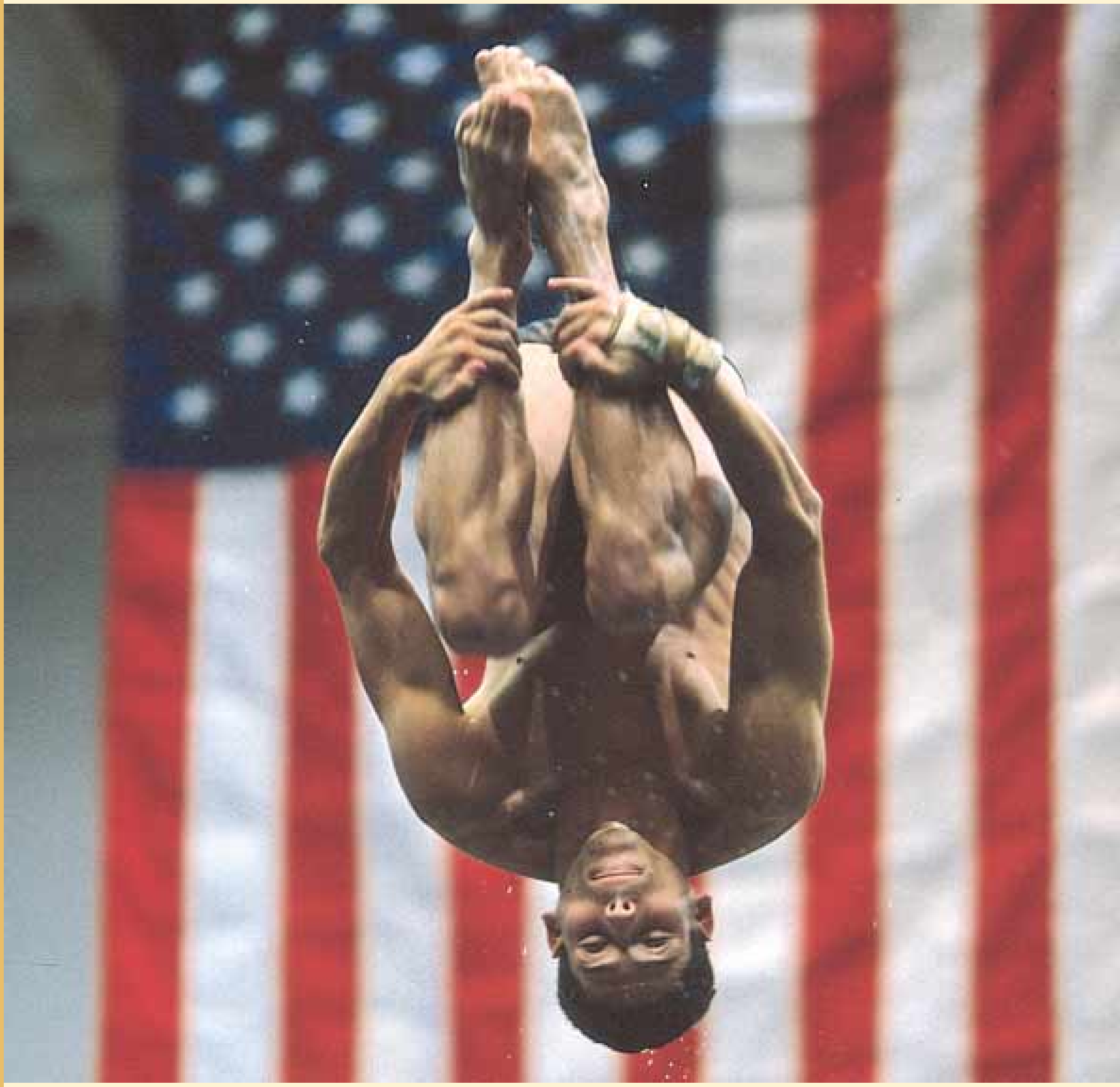
Chapter 11

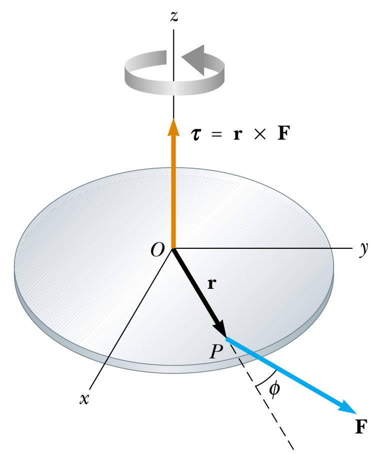
Angular Momentum

CHAPTER OUTLINE



▲ *Mark Ruiz undergoes a rotation during a dive at the U.S. Olympic trials in June 2000. He spins at a higher rate when he curls up and grabs his ankles due to the principle of conservation of angular momentum, as discussed in this chapter. (Otto Greule/Allsport/Getty)*

The central topic of this chapter is angular momentum, a quantity that plays a key role in rotational dynamics. In analogy to the principle of conservation of linear momentum, we find that the angular momentum of a system is conserved if no external torques act on the system. Like the law of conservation of linear momentum, the law of conservation of angular momentum is a fundamental law of physics, equally valid for relativistic and quantum systems.



Active Figure 11.1 The torque vector $\boldsymbol{\tau}$ lies in a direction perpendicular to the plane formed by the position vector $\mathbf{r}$ and the applied force vector $\mathbf{F}$.

At the Active Figures link at http://www.pse6.com, you can move point P and change the force vector F to see the effect on the torque vector.

11.1 The Vector Product and Torque

An important consideration in defining angular momentum is the process of multiplying two vectors by means of the operation called the *vector product.* We will introduce the vector product by considering torque as introduced in the preceding chapter.

Consider a force $\mathbf{F}$ acting on a rigid object at the vector position $\mathbf{r}$ (Fig. 11.1). As we saw in Section 10.6, the *magnitude* of the torque due to this force relative to the origin is $rF \sin \phi$, where ϕ is the angle between $\mathbf{r}$ and $\mathbf{F}$. The axis about which $\mathbf{F}$ tends to produce rotation is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{F}$.

The torque vector $\boldsymbol{\tau}$ is related to the two vectors $\mathbf{r}$ and $\mathbf{F}$. We can establish a mathematical relationship between $\boldsymbol{\tau}$, $\mathbf{r}$, and $\mathbf{F}$ using a mathematical operation called the **vector product,** or **cross product:**

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \tag{11.1}$$

We now give a formal definition of the vector product. Given any two vectors $\mathbf{A}$ and $\mathbf{B}$, the **vector product $\mathbf{A} \times \mathbf{B}$** is defined as a third vector $\mathbf{C}$, which has a magnitude of $AB \sin \theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. That is, if $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{11.2}$$

then its magnitude is

$$C \equiv AB \sin \theta \tag{11.3}$$

The quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$, as shown in Figure 11.2. The *direction* of $\mathbf{C}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$, and the best way to determine this direction is to use the right-hand rule illustrated in Figure 11.2. The four fingers of the right hand are pointed along $\mathbf{A}$ and then "wrapped" into $\mathbf{B}$ through the angle θ. The direction of the upright thumb is the direction of $\mathbf{A} \times \mathbf{B} = \mathbf{C}$. Because of the notation, $\mathbf{A} \times \mathbf{B}$ is often read "$\mathbf{A}$ cross $\mathbf{B}$"; hence, the term *cross product.*

PITFALL PREVENTION

11.1 The Cross Product is a Vector

Remember that the result of taking a cross product between two vectors is *a third vector.* Equation 11.3 gives only the magnitude of this vector.

Some properties of the vector product that follow from its definition are as follows:

1. Unlike the scalar product, the vector product is *not* commutative. Instead, the order in which the two vectors are multiplied in a cross product is important:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{11.4}$$

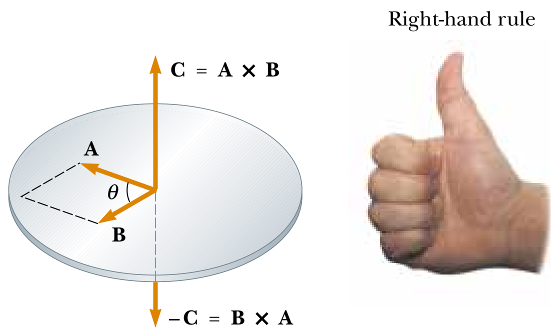


Figure 11.2 The vector product $\mathbf{A} \times \mathbf{B}$ is a third vector $\mathbf{C}$ having a magnitude $AB \sin \theta$ equal to the area of the parallelogram shown. The direction of $\mathbf{C}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$, and this direction is determined by the right-hand rule.

Therefore, if you change the order of the vectors in a cross product, you must change the sign. You can easily verify this relationship with the right-hand rule.

2. If $\mathbf{A}$ is parallel to $\mathbf{B}$ ($\theta = 0°$ or $180°$), then $\mathbf{A} \times \mathbf{B} = 0$; therefore, it follows that $\mathbf{A} \times \mathbf{A} = 0$.
3. If $\mathbf{A}$ is perpendicular to $\mathbf{B}$, then $|\mathbf{A} \times \mathbf{B}| = AB$.
4. The vector product obeys the distributive law:

Properties of the vector product

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{11.5}$$

5. The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \tag{11.6}$$

where it is important to preserve the multiplicative order of $\mathbf{A}$ and $\mathbf{B}$, in view of Equation 11.4.

It is left as an exercise (Problem 10) to show from Equations 11.3 and 11.4 and from the definition of unit vectors that the cross products of the rectangular unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ obey the following rules:

Cross products of unit vectors

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \tag{11.7a}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \hat{\mathbf{k}} \tag{11.7b}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}} \tag{11.7c}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}} \tag{11.7d}$$

Signs are interchangeable in cross products. For example, $\mathbf{A} \times (-\mathbf{B}) = -\mathbf{A} \times \mathbf{B}$ and $\hat{\mathbf{i}} \times (-\hat{\mathbf{j}}) = -\hat{\mathbf{i}} \times \hat{\mathbf{j}}$.

The cross product of any two vectors $\mathbf{A}$ and $\mathbf{B}$ can be expressed in the following determinant form:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{\mathbf{k}}$$

Expanding these determinants gives the result

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} - (A_xB_z - A_zB_x)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}} \tag{11.8}$$

Given the definition of the cross product, we can now assign a direction to the torque vector. If the force lies in the xy plane, as in Figure 11.1, the torque $\boldsymbol{\tau}$ is represented by a vector parallel to the z axis. The force in Figure 11.1 creates a torque that tends to rotate the object counterclockwise about the z axis; thus the direction of $\boldsymbol{\tau}$ is toward increasing z, and $\boldsymbol{\tau}$ is therefore in the positive z direction. If we reversed the direction of $\mathbf{F}$ in Figure 11.1, then $\boldsymbol{\tau}$ would be in the negative z direction.

Quick Quiz 11.1 Which of the following is equivalent to the following *scalar* product: $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{A})$? (a) $\mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A}$ (b) $(\mathbf{A} \times \mathbf{A}) \cdot (\mathbf{B} \times \mathbf{B})$ (c) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$ (d) $-(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$

Quick Quiz 11.2 Which of the following statements is true about the relationship between the magnitude of the cross product of two vectors and the product of the magnitudes of the vectors? (a) $|\mathbf{A} \times \mathbf{B}|$ is larger than AB; (b) $|\mathbf{A} \times \mathbf{B}|$ is smaller than AB; (c) $|\mathbf{A} \times \mathbf{B}|$ could be larger or smaller than AB, depending on the angle between the vectors; (d) $|\mathbf{A} \times \mathbf{B}|$ could be equal to AB.

Example 11.1 The Vector Product

Two vectors lying in the xy plane are given by the equations $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\mathbf{B} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. Find $\mathbf{A} \times \mathbf{B}$ and verify that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Solution Using Equations 11.7a through 11.7d, we obtain

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \\ &= 2\hat{\mathbf{i}} \times 2\hat{\mathbf{j}} + 3\hat{\mathbf{j}} \times (-\hat{\mathbf{i}}) = 4\hat{\mathbf{k}} + 3\hat{\mathbf{k}} = \boxed{7\hat{\mathbf{k}}}\end{aligned}$$

(We have omitted the terms containing $\hat{\mathbf{i}} \times \hat{\mathbf{i}}$ and $\hat{\mathbf{j}} \times \hat{\mathbf{j}}$ because, as Equation 11.7a shows, they are equal to zero.)

We can show that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, because

$$\begin{aligned}\mathbf{B} \times \mathbf{A} &= (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \times (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \\ &= -\hat{\mathbf{i}} \times 3\hat{\mathbf{j}} + 2\hat{\mathbf{j}} \times 2\hat{\mathbf{i}} = -3\hat{\mathbf{k}} - 4\hat{\mathbf{k}} = \boxed{-7\hat{\mathbf{k}}}\end{aligned}$$

Therefore, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

As an alternative method for finding $\mathbf{A} \times \mathbf{B}$, we could use Equation 11.8, with $A_x = 2$, $A_y = 3$, $A_z = 0$ and $B_x = -1$, $B_y = 2$, $B_z = 0$:

$$\mathbf{A} \times \mathbf{B} = (0)\hat{\mathbf{i}} - (0)\hat{\mathbf{j}} + [(2)(2) - (3)(-1)]\hat{\mathbf{k}} = 7\hat{\mathbf{k}}$$

Example 11.2 The Torque Vector

A force of $\mathbf{F} = (2.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})$ N is applied to an object that is pivoted about a fixed axis aligned along the z coordinate axis. If the force is applied at a point located at $\mathbf{r} = (4.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}})$ m, find the torque vector $\boldsymbol{\tau}$.

Solution The torque vector is defined by means of a cross product in Equation 11.1:

$$\begin{aligned}\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} &= [(4.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}})\ \text{m}] \\ &\quad \times [(2.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})\ \text{N}] \\ &= [(4.00)(2.00)\hat{\mathbf{i}} \times \hat{\mathbf{i}} + (4.00)(3.00)\hat{\mathbf{i}} \times \hat{\mathbf{j}} \\ &\quad + (5.00)(2.00)\hat{\mathbf{j}} \times \hat{\mathbf{i}} \\ &\quad + (5.00)(3.00)\hat{\mathbf{j}} \times \hat{\mathbf{j}}]\ \text{N}\cdot\text{m} \\ &= [12.0\hat{\mathbf{i}} \times \hat{\mathbf{j}} + 10.0\hat{\mathbf{j}} \times \hat{\mathbf{i}}]\ \text{N}\cdot\text{m} \\ &= [12.0\hat{\mathbf{k}} - 10.0\hat{\mathbf{k}})\ \text{N}\cdot\text{m} \\ &= \boxed{2.0\hat{\mathbf{k}}\ \text{N}\cdot\text{m}}\end{aligned}$$

Notice that both $\mathbf{r}$ and $\mathbf{F}$ are in the xy plane. As expected, the torque vector is perpendicular to this plane, having only a z component.

11.2 Angular Momentum

Imagine a rigid pole sticking up through the ice on a frozen pond (Fig. 11.3). A skater glides rapidly toward the pole, aiming a little to the side so that she does not hit it. As she approaches the pole, she reaches out and grabs it, an action that causes her to move in a circular path around the pole. Just as the idea of linear momentum helps us analyze translational motion, a rotational analog—*angular momentum*—helps us analyze the motion of this skater and other objects undergoing rotational motion.

Active Figure 11.3 As the skater passes the pole, she grabs hold of it. This causes her to swing around the pole rapidly in a circular path.

At the Active Figures link at http://www.pse6.com, you can change the speed of the skater and her distance to the pole and watch her spin when she grabs the pole.

In Chapter 9, we began by developing the mathematical form of linear momentum and then proceeded to show how this new quantity was valuable in problem-solving. We will follow a similar procedure for angular momentum.

Consider a particle of mass m located at the vector position $\mathbf{r}$ and moving with linear momentum $\mathbf{p}$ as in Figure 11.4. In describing linear motion, we found that the net force on the particle equals the time rate of change of its linear momentum, $\Sigma \mathbf{F} = d\mathbf{p}/dt$ (see Eq. 9.3). Let us take the cross product of each side of Equation 9.3 with $\mathbf{r}$, which gives us the net torque on the particle on the left side of the equation:

$$\mathbf{r} \times \sum \mathbf{F} = \sum \boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

Now let us add to the right-hand side the term $\dfrac{d\mathbf{r}}{dt} \times \mathbf{p}$, which is zero because $d\mathbf{r}/dt = \mathbf{v}$ and $\mathbf{v}$ and $\mathbf{p}$ are parallel. Thus,

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

We recognize the right-hand side of this equation as the derivative of $\mathbf{r} \times \mathbf{p}$ (see Equation 11.6). Therefore,

$$\sum \boldsymbol{\tau} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} \tag{11.9}$$

This looks very similar in form to Equation 9.3, $\Sigma \mathbf{F} = d\mathbf{p}/dt$. This suggests that the combination $\mathbf{r} \times \mathbf{p}$ should play the same role in rotational motion that $\mathbf{p}$ plays in translational motion. We call this combination the *angular momentum* of the particle:

Angular momentum of a particle

The instantaneous **angular momentum L** of a particle relative to the origin O is defined by the cross product of the particle's instantaneous position vector $\mathbf{r}$ and its instantaneous linear momentum $\mathbf{p}$:

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{11.10}$$

This allows us to write Equation 11.9 as

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{11.11}$$

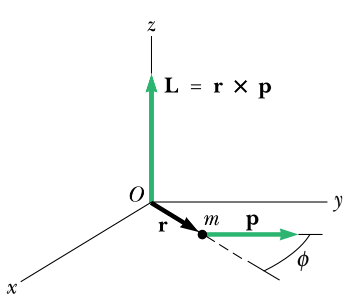
which is the rotational analog of Newton's second law, $\Sigma \mathbf{F} = d\mathbf{p}/dt$. Note that torque causes the angular momentum $\mathbf{L}$ to change just as force causes linear momentum $\mathbf{p}$ to change. Equation 11.11 states that **the torque acting on a particle is equal to the time rate of change of the particle's angular momentum.**

Note that Equation 11.11 is valid only if $\Sigma \boldsymbol{\tau}$ and $\mathbf{L}$ are measured about the same origin. (Of course, the same origin must be used in calculating all of the torques.) Furthermore, **the expression is valid for any origin fixed in an inertial frame.**

The SI unit of angular momentum is $\text{kg} \cdot \text{m}^2/\text{s}$. Note also that both the magnitude and the direction of $\mathbf{L}$ depend on the choice of origin. Following the right-hand rule, we see that the direction of $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$. In Figure 11.4, $\mathbf{r}$ and $\mathbf{p}$ are in the xy plane, and so $\mathbf{L}$ points in the z direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of $\mathbf{L}$ is

$$L = mvr \sin \phi \tag{11.12}$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{p}$. It follows that L is zero when $\mathbf{r}$ is parallel to $\mathbf{p}$ ($\phi = 0$ or $180°$). In other words, when the linear velocity of the particle is along a line that passes through the origin, the particle has zero angular momentum with respect to the origin. On the other hand, if $\mathbf{r}$ is perpendicular to $\mathbf{p}$ ($\phi = 90°$), then $L = mvr$. At that instant, the particle moves exactly as if it were on the rim of a wheel rotating about the origin in a plane defined by $\mathbf{r}$ and $\mathbf{p}$.



Active Figure 11.4 The angular momentum $\mathbf{L}$ of a particle of mass m and linear momentum $\mathbf{p}$ located at the vector position $\mathbf{r}$ is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of $\mathbf{L}$ depends on the origin about which it is measured and is a vector perpendicular to both $\mathbf{r}$ and $\mathbf{p}$.

At the Active Figures link at http://www.pse6.com, you can change the position vector r and the momentum vector p to see the effect on the angular momentum vector.

Quick Quiz 11.3 Recall the skater described at the beginning of this section. Let her mass be m. What would be her angular momentum relative to the pole at the instant she is a distance d from the pole if she were skating directly toward it at speed v? (a) zero (b) mvd (c) impossible to determine

Quick Quiz 11.4 Consider again the skater in Quick Quiz 11.3. What would be her angular momentum relative to the pole at the instant she is a distance d from the pole if she were skating at speed v along a straight line that would pass within a distance a from the pole? (a) zero (b) mvd (c) mva (d) impossible to determine

PITFALL PREVENTION

11.2 Is Rotation Necessary for Angular Momentum?

Notice that we can define angular momentum even if the particle is not moving in a circular path. Even a particle moving in a straight line has angular momentum about any axis displaced from the path of the particle.

Example 11.3 Angular Momentum of a Particle in Circular Motion

A particle moves in the xy plane in a circular path of radius r, as shown in Figure 11.5. Find the magnitude and direction of its angular momentum relative to O when its linear velocity is $\mathbf{v}$.

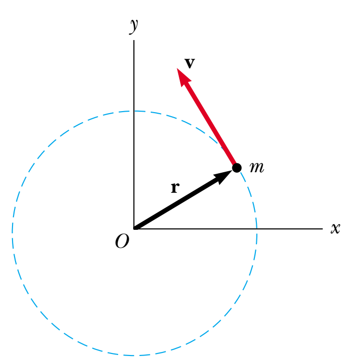


Figure 11.5 (Example 11.3) A particle moving in a circle of radius r has an angular momentum about O that has magnitude mvr. The vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ points *out* of the diagram.

Solution The linear momentum of the particle is always changing (in direction, not magnitude). You might be tempted, therefore, to conclude that the angular momentum of the particle is always changing. In this situation, however, this is not the case—let us see why. From Equation 11.12, the magnitude of $\mathbf{L}$ is given by

$$L = mvr \sin 90° = \boxed{mvr}$$

where we have used $\phi = 90°$ because $\mathbf{v}$ is perpendicular to $\mathbf{r}$. This value of L is constant because all three factors on the right are constant.

The direction of $\mathbf{L}$ also is constant, even though the direction of $\mathbf{p} = m\mathbf{v}$ keeps changing. You can visualize this by applying the right-hand rule to find the direction of $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$ in Figure 11.5. Your thumb points upward and away from the page; this is the direction of $\mathbf{L}$ Hence, we can write the vector expression $\mathbf{L} = (mvr)\hat{\mathbf{k}}$. If the particle were to move clockwise, $\mathbf{L}$ would point downward and into the page. **A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.**

Angular Momentum of a System of Particles

In Section 9.6, we showed that Newton's second law for a particle could be extended to a system of particles, resulting in:

$$\sum \mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}_{\text{tot}}}{dt}$$

This equation states that the net external force on a system of particles is equal to the time rate of change of the total linear momentum of the system. Let us see if there is a similar statement that can be made in rotational motion. The total angular momentum of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L}_{\text{tot}} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

where the vector sum is over all n particles in the system.

Let us differentiate this equation with respect to time:

$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \boldsymbol{\tau}_i$$

where we have used Equation 11.11 to replace the time rate of change of the angular momentum of each particle with the net torque on the particle.

The torques acting on the particles of the system are those associated with internal forces between particles and those associated with external forces. However, the net torque associated with all internal forces is zero. To understand this, recall that Newton's third law tells us that internal forces between particles of the system are equal in magnitude and opposite in direction. If we assume that these forces lie along the line of separation of each pair of particles, then the total torque around some axis passing through an origin O due to each action–reaction force pair is zero. That is, the moment arm d from O to the line of action of the forces is equal for both particles and the forces are in opposite directions. In the summation, therefore, we see that the net internal torque vanishes. We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system, so that we have

The net external torque on a system equals the time rate of change of angular momentum of the system

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}_{\text{tot}}}{dt} \tag{11.13}$$

That is

> the net external torque acting on a system about some axis passing through an origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin.

Note that Equation 11.13 is indeed the rotational analog of $\Sigma\mathbf{F}_{\text{ext}} = d\mathbf{p}_{\text{tot}}/dt$, for a system of particles.

Although we do not prove it here, the following statement is an important theorem concerning the angular momentum of a system relative to the system's center of mass:

> The resultant torque acting on a system about an axis through the center of mass equals the time rate of change of angular momentum of the system regardless of the motion of the center of mass.

This theorem applies even if the center of mass is accelerating, provided $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

Example 11.4 Two Connected Objects

A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure 11.6. The radius of the pulley is R, and the mass of the rim is M. The spokes of the pulley have negligible mass. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

Solution We need to determine the angular momentum of the system, which consists of the two objects and the pulley.

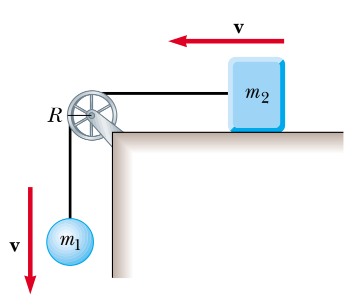


Figure 11.6 (Example 11.4) When the system is released, the sphere moves downward and the block moves to the left.

Let us calculate the angular momentum about an axis that coincides with the axle of the pulley. The angular momentum of the system includes that of two objects moving translationally (the sphere and the block) and one object undergoing pure rotation (the pulley).

At any instant of time, the sphere and the block have a common speed v, so the angular momentum of the sphere is m_1vR, and that of the block is m_2vR. At the same instant, all points on the rim of the pulley also move with speed v, so the angular momentum of the pulley is MvR. Hence, the total angular momentum of the system is

$$(1) \qquad L = m_1vR + m_2vR + MvR = (m_1 + m_2 + M)vR$$

Now let us evaluate the total external torque acting on the system about the pulley axle. Because it has a moment arm of zero, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block is balanced by the gravitational force $m_2\mathbf{g}$, and so these forces do not contribute to the torque. The gravitational force $m_1\mathbf{g}$ acting on the sphere produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This is the total external torque about the pulley axle;

that is, $\Sigma\tau_{\text{ext}} = m_1gR$. Using this result, together with Equation (1) and Equation 11.13, we find

$$\sum \tau_{\text{ext}} = \frac{dL}{dt}$$

$$m_1gR = \frac{d}{dt}[(m_1 + m_2 + M)vR]$$

$$(2) \qquad m_1gR = (m_1 + m_2 + M)R\,\frac{dv}{dt}$$

Because $dv/dt = a$, we can solve this for a to obtain

$$a = \frac{m_1g}{m_1 + m_2 + M}$$

You may wonder why we did not include the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration, and we analyzed the system as a whole. Only *external* torques contribute to the change in the system's angular momentum.

11.3 Angular Momentum of a Rotating Rigid Object

In Example 11.4, we considered the angular momentum of a deformable system. Let us now restrict our attention to a nondeformable system—a rigid object. Consider a rigid object rotating about a fixed axis that coincides with the z axis of a coordinate system, as shown in Figure 11.7. Let us determine the angular momentum of this object. Each *particle* of the object rotates in the xy plane about the z axis with an angular speed ω. The magnitude of the angular momentum of a particle of mass m_i about the z axis is $m_iv_ir_i$. Because $v_i = r_i\omega$, we can express the magnitude of the angular momentum of this particle as

$$L_i = m_ir_i^2\omega$$

The vector $\mathbf{L}_i$ is directed along the z axis, as is the vector $\boldsymbol{\omega}$.

We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of L_i over all particles:

$$L_z = \sum_i L_i = \sum_i m_ir_i^2\omega = \left(\sum_i m_ir_i^2\right)\omega$$

$$L_z = I\omega \qquad (11.14)$$

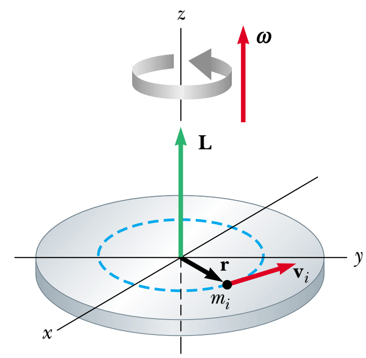


Figure 11.7 When a rigid object rotates about an axis, the angular momentum $\mathbf{L}$ is in the same direction as the angular velocity $\boldsymbol{\omega}$, according to the expression $\mathbf{L} = I\boldsymbol{\omega}$.

where we have recognized $\sum_i m_ir_i^2$ as the moment of inertia I of the object about the z axis (Equation 10.15).

Now let us differentiate Equation 11.14 with respect to time, noting that I is constant for a rigid object:

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \qquad (11.15)$$

where α is the angular acceleration relative to the axis of rotation. Because dL_z/dt is equal to the net external torque (see Eq. 11.13), we can express Equation 11.15 as

$$\sum \tau_{\text{ext}} = I\alpha \qquad (11.16)$$

Rotational form of Newton's second law

That is, the net external torque acting on a rigid object rotating about a fixed axis equals the moment of inertia about the rotation axis multiplied by the object's angular acceleration relative to that axis. This result is the same as Equation 10.21, which was derived using a force approach, but we derived Equation 11.16 using the concept of angular momentum. This equation is also valid for a rigid object rotating about a moving axis provided the moving axis (1) passes through the center of mass and (2) is a symmetry axis.

If a symmetrical object rotates about a fixed axis passing through its center of mass, you can write Equation 11.14 in vector form as $\mathbf{L} = I\boldsymbol{\omega}$, where $\mathbf{L}$ is the total angular momentum of the object measured with respect to the axis of rotation. Furthermore, the expression is valid for any object, regardless of its symmetry, if **L** stands for the component of angular momentum along the axis of rotation.[1]

Quick Quiz 11.5 A solid sphere and a hollow sphere have the same mass and radius. They are rotating with the same angular speed. The one with the higher angular momentum is (a) the solid sphere (b) the hollow sphere (c) they both have the same angular momentum (d) impossible to determine.

Example 11.5 Bowling Ball

Estimate the magnitude of the angular momentum of a bowling ball spinning at 10 rev/s, as shown in Figure 11.8.

Solution We start by making some estimates of the relevant physical parameters and model the ball as a uniform solid sphere. A typical bowling ball might have a mass of 6.0 kg and a radius of 12 cm. The moment of inertia of a solid sphere about an axis through its center is, from Table 10.2,

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5}(6.0\text{ kg})(0.12\text{ m})^2 = 0.035\text{ kg}\cdot\text{m}^2$$

Therefore, the magnitude of the angular momentum is

$$\begin{aligned} L_z = I\omega &= (0.035\text{ kg}\cdot\text{m}^2)(10\text{ rev/s})(2\pi\text{ rad/rev}) \\ &= 2.2\text{ kg}\cdot\text{m}^2\text{/s} \end{aligned}$$

Because of the roughness of our estimates, we probably want to keep only one significant figure, and so $L_z \approx \boxed{2\text{ kg}\cdot\text{m}^2\text{/s.}}$

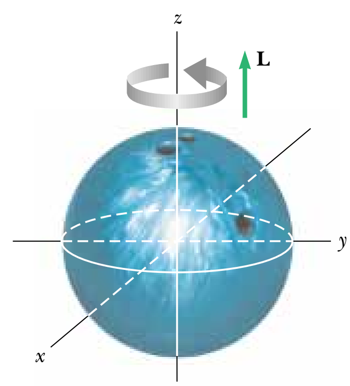


Figure 11.8 (Example 11.5) A bowling ball that rotates about the *z* axis in the direction shown has an angular momentum **L** in the positive *z* direction. If the direction of rotation is reversed, **L** points in the negative *z* direction.

Example 11.6 The Seesaw

Interactive

A father of mass m_f and his daughter of mass m_d sit on opposite ends of a seesaw at equal distances from the pivot at the center (Fig. 11.9). The seesaw is modeled as a rigid rod of mass M and length ℓ and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed ω.

(A) Find an expression for the magnitude of the system's angular momentum.

Solution The moment of inertia of the system equals the sum of the moments of inertia of the three components: the seesaw and the two individuals, whom we will model as particles. Referring to Table 10.2 to obtain the expression for the moment of inertia of the rod, and using the expression $I = mr^2$ for each person, we find that the total moment of inertia about the *z* axis through O is

$$I = \tfrac{1}{12}M\ell^2 + m_f\left(\frac{\ell}{2}\right)^2 + m_d\left(\frac{\ell}{2}\right)^2 = \frac{\ell^2}{4}\left(\frac{M}{3} + m_f + m_d\right)$$

Therefore, the magnitude of the angular momentum is

$$L = I\omega = \boxed{\frac{\ell^2}{4}\left(\frac{M}{3} + m_f + m_d\right)\omega}$$

(B) Find an expression for the magnitude of the angular acceleration of the system when the seesaw makes an angle θ with the horizontal.

Solution To find the angular acceleration of the system at any angle θ, we first calculate the net torque on the system and then use $\Sigma\tau_{\text{ext}} = I\alpha$ to obtain an expression for α.

The torque due to the force $m_f g$ about the pivot is

$$\tau_f = m_f g\,\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_f \text{ out of page})$$

The torque due to the force $m_d g$ about the pivot is

$$\tau_d = -\,m_d g\,\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_d \text{ into page})$$

[1] In general, the expression $\mathbf{L} = I\boldsymbol{\omega}$ is not always valid. If a rigid object rotates about an *arbitrary* axis, **L** and $\boldsymbol{\omega}$ may point in different directions. In this case, the moment of inertia cannot be treated as a scalar. Strictly speaking, $\mathbf{L} = I\boldsymbol{\omega}$ applies only to rigid objects of any shape that rotate about one of three mutually perpendicular axes (called *principal axes*) through the center of mass. This is discussed in more advanced texts on mechanics.

Hence, the net torque exerted on the system about O is

$$\sum \tau_{\text{ext}} = \tau_f + \tau_d = \tfrac{1}{2}(m_f - m_d)g\ell \cos\theta$$

To find α, we use $\Sigma\tau_{\text{ext}} = I\alpha$, where I was obtained in part (A):

$$\alpha = \frac{\sum \tau_{\text{ext}}}{I} = \boxed{\frac{2(m_f - m_d)g\cos\theta}{\ell\left(\frac{M}{3} + m_f + m_d\right)}}$$

Generally, fathers are more massive than daughters, so the angular acceleration is positive. If the seesaw begins in a horizontal orientation ($\theta = 0$) and is released, the rotation will be counterclockwise in Figure 11.9 and the father's end of the seesaw drops. This is consistent with everyday experience.

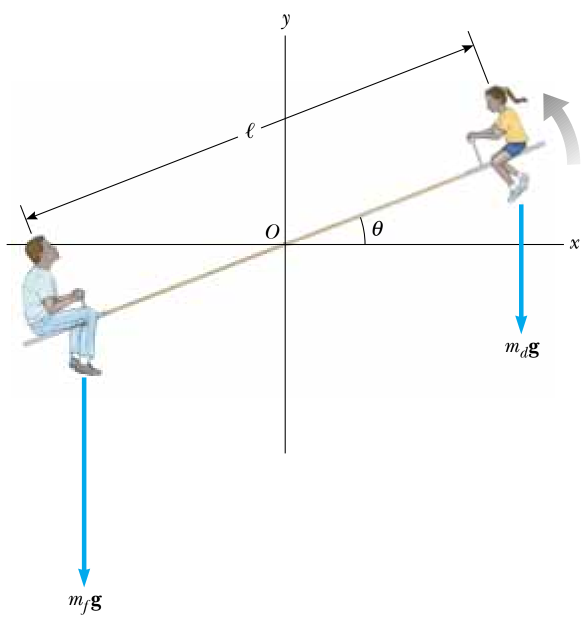


Figure 11.9 (Example 11.6) A father and daughter demonstrate angular momentum on a seesaw.

What If? **After several complaints from the daughter that she simply rises into the air rather than moving up and down as planned, the father moves inward on the seesaw to try to balance the two sides. He moves in to a position that is a distance d from the pivot. What is the angular acceleration of the system in this case when it is released from an arbitrary angle θ?**

Answer The angular acceleration of the system should decrease if the system is more balanced. As the father continues to slide inward, he should reach a point at which the seesaw is balanced and there is no angular acceleration of the system when released.

The total moment of inertia about the z axis through O for the modified system is

$$I = \tfrac{1}{12}M\ell^2 + m_f d^2 + m_d\left(\frac{\ell}{2}\right)^2$$

$$= \frac{\ell^2}{4}\left(\frac{M}{3} + m_d\right) + m_f d^2$$

The net torque exerted on the system about O is

$$\tau_{\text{net}} = \tau_f + \tau_d = m_f g d\cos\theta - \tfrac{1}{2}m_d g\ell\cos\theta$$

Now, the angular acceleration of the system is

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{m_f g d\cos\theta - \frac{1}{2}m_d g\ell\cos\theta}{\frac{\ell^2}{4}[(M/3) + m_d] + m_f d^2}$$

The seesaw will be balanced when the angular acceleration is zero. In this situation, both father and daughter can push off the ground and rise to the highest possible point. We find the required position of the father by setting $\alpha = 0$:

$$\alpha = \frac{m_f g d\cos\theta - \frac{1}{2}m_d g\ell\cos\theta}{(\ell^2/4)[(M/3) + m_d] + m_f d^2} = 0$$

$$m_f g d\cos\theta - \tfrac{1}{2}m_d g\ell\cos\theta = 0$$

$$d = \left(\frac{m_d}{m_f}\right)\tfrac{1}{2}\ell$$

In the rare case that the father and daughter have the same mass, the father is located at the end of the seesaw, $d = \ell/2$.

At the Interactive Worked Example link at **http://www.pse6.com,** *you can move the father and daughter to see the effect on the motion of the system.*

11.4 Conservation of Angular Momentum

In Chapter 9 we found that the total linear momentum of a system of particles remains constant if the system is isolated, that is, if the resultant external force acting on the system is zero. We have an analogous conservation law in rotational motion:

Conservation of angular momentum

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero, that is, if the system is isolated.

Figure 11.10 Figure skater Todd Eldridge is demonstrating angular momentum conservation. When he pulls his arms toward his body, he spins faster.

This follows directly from Equation 11.13, which indicates that if

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}_{\text{tot}}}{dt} = 0 \tag{11.17}$$

then

$$\mathbf{L}_{\text{tot}} = \text{constant} \qquad \text{or} \qquad \mathbf{L}_i = \mathbf{L}_f \tag{11.18}$$

For an isolated system consisting of a number of particles, we write this conservation law as $\mathbf{L}_{\text{tot}} = \Sigma \mathbf{L}_n =$ constant, where the index n denotes the nth particle in the system.

If the mass of an isolated rotating system undergoes redistribution in some way, the system's moment of inertia changes. Because the magnitude of the angular momentum of the system is $L = I\omega$ (Eq. 11.14), conservation of angular momentum requires that the product of I and ω must remain constant. Thus, a change in I for an isolated system requires a change in ω. In this case, we can express the principle of conservation of angular momentum as

$$I_i\omega_i = I_f\omega_f = \text{constant} \tag{11.19}$$

This expression is valid both for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system as long as that axis remains fixed in direction. We require only that the net external torque be zero.

There are many examples that demonstrate conservation of angular momentum for a deformable system. You may have observed a figure skater spinning in the finale of a program (Fig. 11.10). The angular speed of the skater increases when the skater pulls his hands and feet close to his body, thereby decreasing I. Neglecting friction between skates and ice, no external torques act on the skater. Because the angular momentum of the skater is conserved, the product $I\omega$ remains constant, and a decrease in the moment of inertia of the skater causes an increase in the angular speed. Similarly, when divers or acrobats wish to make several somersaults, they pull their hands and feet close to their bodies to rotate at a higher rate, as in the opening photograph of this chapter. In these cases, the external force due to gravity acts through the center of mass and hence exerts no torque about this point. Therefore, the angular momentum about the center of mass must be conserved—that is, $I_i\omega_i = I_f\omega_f$. For example, when divers wish to double their angular speed, they must reduce their moment of inertia to half its initial value.

In Equation 11.18 we have a third conservation law to add to our list. We can now state that the energy, linear momentum, and angular momentum of an isolated system all remain constant:

$$\left.\begin{aligned} E_i &= E_f \\ \mathbf{p}_i &= \mathbf{p}_f \\ \mathbf{L}_i &= \mathbf{L}_f \end{aligned}\right\} \quad \text{For an isolated system}$$

Quick Quiz 11.6 A competitive diver leaves the diving board and falls toward the water with her body straight and rotating slowly. She pulls her arms and legs into a tight tuck position. Her angular speed (a) increases (b) decreases (c) stays the same (d) is impossible to determine.

Quick Quiz 11.7 Consider the competitive diver in Quick Quiz 11.6 again. When she goes into the tuck position, the rotational kinetic energy of her body (a) increases (b) decreases (c) stays the same (d) is impossible to determine.

Example 11.7 Formation of a Neutron Star

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0 km. Determine the period of rotation of the neutron star.

Solution The same physics that makes a skater spin faster with his arms pulled in describes the motion of the neutron star. Let us assume that during the collapse of the stellar core, (1) no external torque acts on it, (2) it remains spherical with the same relative mass distribution, and (3) its mass remains constant. Also, let us use the symbol T for the period, with T_i being the initial period of the star and T_f being the period of the neutron star. The period is the length of time a point on the star's equator takes to make one complete circle around the axis of rotation. The angular speed of the star is given by $\omega = 2\pi/T$. Therefore, Equation 11.19 gives

$$I_i\omega_i = I_f\omega_f$$

$$I_i\left(\frac{2\pi}{T_i}\right) = I_f\left(\frac{2\pi}{T_f}\right)$$

We don't know the mass distribution of the star, but we have assumed that the distribution is symmetric, so that the moment of inertia can be expressed as kMR^2, where k is some numerical constant. (From Table 10.2, for example, we see that $k = 2/5$ for a solid sphere and $k = 2/3$ for a spherical shell.) Thus, we can rewrite the preceding equation as

$$kMR_i^2\left(\frac{2\pi}{T_i}\right) = kMR_f^2\left(\frac{2\pi}{T_f}\right)$$

$$T_f = \left(\frac{R_f^2}{R_i^2}\right)T_i$$

Substituting numerical values gives

$$T_f = (30\text{ days})\left(\frac{3.0\text{ km}}{1.0 \times 10^4\text{ km}}\right)^2 = 2.7 \times 10^{-6}\text{ days}$$

$$= \boxed{0.23\text{ s}}$$

Thus, the neutron star rotates about four times each second.

Example 11.8 The Merry-Go-Round

A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless vertical axle (Fig. 11.11). The platform has a mass $M = 100$ kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when he reaches a point $r = 0.50$ m from the center?

Solution The speed change here is similar to the increase in angular speed of the spinning skater when he pulls his arms inward. Let us denote the moment of inertia of the platform as I_p and that of the student as I_s. Modeling the student as a particle, we can write the initial moment of inertia I_i of the system (student plus platform) about the axis of rotation:

$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

When the student walks to the position $r < R$, the moment of inertia of the system reduces to

$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

Note that we still use the greater radius R when calculating I_{pf} because the radius of the platform does not change. Because no external torques act on the system about the axis of rotation, we can apply the law of conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f$$

$$(\tfrac{1}{2}MR^2 + mR^2)\omega_i = (\tfrac{1}{2}MR^2 + mr^2)\omega_f$$

$$\omega_f = \left(\frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$= \left(\frac{\frac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(2.0\text{ m})^2}{\frac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(0.50\text{ m})^2}\right)(2.0\text{ rad/s})$$

$$= \left(\frac{440\text{ kg}\cdot\text{m}^2}{215\text{ kg}\cdot\text{m}^2}\right)(2.0\text{ rad/s}) = \boxed{4.1\text{ rad/s}}$$

As expected, the angular speed increases.

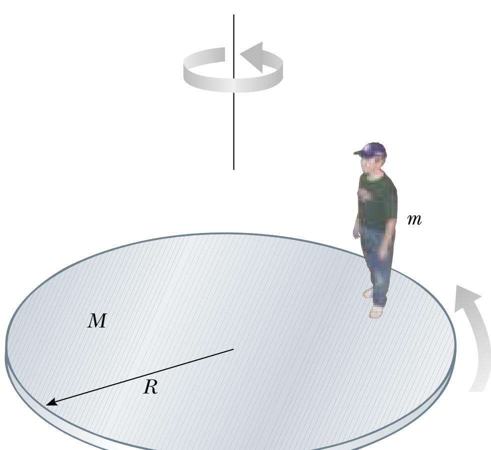


Figure 11.11 (Example 11.8) As the student walks toward the center of the rotating platform, the angular speed of the system increases because the angular momentum of the system remains constant.

What If? **What if we were to measure the kinetic energy of the system before and after the student walks inward? Are they the same?**

Answer You may be tempted to say yes because the system is isolated. But remember that energy comes in several forms, so we have to handle an energy question carefully. The initial kinetic energy is

$$K_i = \tfrac{1}{2}I_i\omega_i^2 = \tfrac{1}{2}(440\ \text{kg}\cdot\text{m}^2)(2.0\ \text{rad/s})^2 = 880\ \text{J}$$

The final kinetic energy is

$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}(215\ \text{kg}\cdot\text{m}^2)(4.1\ \text{rad/s})^2 = 1.81 \times 10^3\ \text{J}$$

Thus, the kinetic energy of the system *increases*. The student must do work to move himself closer to the center of rotation, so this extra kinetic energy comes from chemical potential energy in the body of the student.

Example 11.9 The Spinning Bicycle Wheel

In a favorite classroom demonstration, a student holds the axle of a spinning bicycle wheel while seated on a stool that is free to rotate (Fig. 11.12). The student and stool are initially at rest while the wheel is spinning in a horizontal plane with an initial angular momentum $\mathbf{L}_i$ that points upward. When the wheel is inverted about its center by 180°, the student and stool start rotating. In terms of $\mathbf{L}_i$, what are the magnitude and the direction of $\mathbf{L}$ for the student plus stool?

Figure 11.12 (Example 11.9) The wheel is initially spinning when the student is at rest. What happens when the wheel is inverted?

Solution The system consists of the student, the wheel, and the stool. Initially, the total angular momentum of the system $\mathbf{L}_i$ comes entirely from the spinning wheel. As the wheel is inverted, the student applies a torque to the wheel, but this torque is internal to the system. No external torque is acting on the system about the vertical axis. Therefore, the angular momentum of the system is conserved. Initially, we have

$$\mathbf{L}_{\text{system}} = \mathbf{L}_i = \mathbf{L}_{\text{wheel}} \qquad \text{(upward)}$$

After the wheel is inverted, we have $\mathbf{L}_{\text{inverted wheel}} = -\mathbf{L}_i$. For angular momentum to be conserved, some other part of the system has to start rotating so that the total final angular momentum equals the initial angular momentum $\mathbf{L}_i$. That other part of the system is the student plus the stool she is sitting on. So, we can now state that

$$\mathbf{L}_f = \mathbf{L}_i = \mathbf{L}_{\text{student+stool}} - \mathbf{L}_i$$

$$\mathbf{L}_{\text{student+stool}} = \boxed{2\mathbf{L}_i}$$

Example 11.10 Disk and Stick

Interactive

A 2.0-kg disk traveling at 3.0 m/s strikes a 1.0-kg stick of length 4.0 m that is lying flat on nearly frictionless ice, as shown in Figure 11.13. Assume that the collision is elastic and that the disk does not deviate from its original line of motion. Find the translational speed of the disk, the translational speed of the stick, and the angular speed of the stick after the collision. The moment of inertia of the stick about its center of mass is 1.33 kg · m^2.

Solution Conceptualize the situation by considering Figure 11.13 and imagining what happens after the disk hits the stick. Because the disk and stick form an isolated system, we can assume that total energy, linear momentum, and angular momentum are all conserved. Thus, we can categorize this as a problem in which all three conservation laws might play a part. To analyze the problem, first note that we have three unknowns, and so we need three equations to solve simultaneously. The first comes from the law of the conservation of linear momentum:

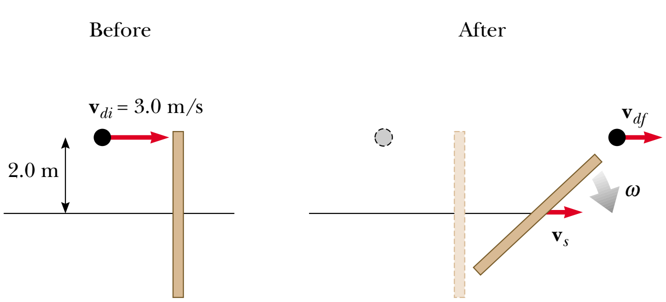


Figure 11.13 (Example 11.10) Overhead view of a disk striking a stick in an elastic collision, which causes the stick to rotate and move to the right.

$$p_i = p_f$$

$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$(2.0\ \text{kg})(3.0\ \text{m/s}) = (2.0\ \text{kg})v_{df} + (1.0\ \text{kg})v_s$$

$$(1) \qquad 6.0\ \text{kg}\cdot\text{m/s} - (2.0\ \text{kg})v_{df} = (1.0\ \text{kg})v_s$$

Now we apply the law of conservation of angular momentum, using the initial position of the center of the stick as our reference point. We know that the component of angular momentum of the disk along the axis perpendicular to the plane of the ice is negative. (The right-hand rule shows that $\mathbf{L}_d$ points into the ice.) Applying conservation of angular momentum to the system gives

$$L_i = L_f$$

$$-rm_d v_{di} = -rm_d v_{df} + I\omega$$

$$-(2.0\text{ m})(2.0\text{ kg})(3.0\text{ m/s}) = -(2.0\text{ m})(2.0\text{ kg})v_{df} + (1.33\text{ kg}\cdot\text{m}^2)\omega$$

$$-12\text{ kg}\cdot\text{m}^2/\text{s} = -(4.0\text{ kg}\cdot\text{m})v_{df} + (1.33\text{ kg}\cdot\text{m}^2)\omega$$

$$(2) \quad -9.0\text{ rad/s} + (3.0\text{ rad/m})v_{df} = \omega$$

We use the fact that radians are dimensionless to ensure consistent units for each term.

Finally, the elastic nature of the collision tells us that kinetic energy is conserved; in this case, the kinetic energy consists of translational and rotational forms:

$$K_i = K_f$$

$$\tfrac{1}{2}m_d v_{di}^2 = \tfrac{1}{2}m_d v_{df}^2 + \tfrac{1}{2}m_s v_s^2 + \tfrac{1}{2}I\omega^2$$

$$\tfrac{1}{2}(2.0\text{ kg})(3.0\text{ m/s})^2 = \tfrac{1}{2}(2.0\text{ kg})v_{df}^2 + \tfrac{1}{2}(1.0\text{ kg})v_s^2 + \tfrac{1}{2}(1.33\text{ kg}\cdot\text{m}^2)\omega^2$$

$$(3) \quad 18\text{ m}^2/\text{s}^2 = 2.0\, v_{df}^2 + v_s^2 + (1.33\text{ m}^2)\omega^2$$

In solving Equations (1), (2), and (3) simultaneously, we find that $v_{df} =$ 2.3 m/s, $v_s =$ 1.3 m/s, and $\omega =$ -2.0 rad/s.

To finalize the problem, note that these values seem reasonable. The disk is moving more slowly after the collision than it was before the collision, and the stick has a small translational speed. Table 11.1 summarizes the initial and final values of variables for the disk and the stick, and verifies the conservation of linear momentum, angular momentum, and kinetic energy.

What If? **What if the collision between the disk and the stick is perfectly inelastic? How does this change the analysis?**

Answer In this case, the disk adheres to the end of the stick upon collision. The conservation of linear momentum principle leading to Equation (1) would be altered to

$$p_i = p_f$$

$$m_d v_{di} = (m_d + m_s)v_{\text{CM}}$$

$$(2.0\text{ kg})(3.0\text{ m/s}) = (2.0\text{ kg} + 1.0\text{ kg})v_{\text{CM}}$$

$$v_{\text{CM}} = 2.0\text{ m/s}$$

For the rotational part of this question, we need to find the center of mass of the system of the disk and the stick. Choosing the center of the stick as the origin, the y position of the center of mass along the vertical stick is

$$y_{\text{CM}} = \frac{(2.0\text{ kg})(2.0\text{ m}) + (1.0\text{ kg})(0)}{(2.0\text{ kg} + 1.0\text{ kg})} = 1.33\text{ m}$$

Thus, the center of mass of the system is $2.0\text{ m} - 1.33\text{ m} = 0.67\text{ m}$ from the upper end of the stick.

The conservation of angular momentum principle leading to Equation (2) would be altered to the following, evaluating angular momenta around the center of mass of the system:

$$L_i = L_f$$

$$-rm_d v_{di} = I_d\omega + I_s\omega$$

$$(4) \quad -(0.67\text{ m})m_d v_{di} = [m_d(0.67\text{ m})^2]\omega + I_s\omega$$

The moment of inertia of the stick around the center of mass *of the system* is found from the parallel-axis theorem:

$$I_s = I_{\text{CM}} + MD^2$$
$$= 1.33\text{ kg}\cdot\text{m}^2 + (1.0\text{ kg})(1.33\text{ m})^2 = 3.1\text{ kg}\cdot\text{m}^2$$

Thus, Equation (4) becomes

$$-(0.67\text{ m})(2.0\text{ kg})(3.0\text{ m/s}) = [(2.0\text{ kg})(0.67\text{ m})^2]\omega + (3.1\text{ kg}\cdot\text{m}^2)\omega$$

$$-4.0\text{ kg}\cdot\text{m}^2/\text{s} = (4.0\text{ kg}\cdot\text{m}^2)\omega$$

$$\omega = \frac{-4.0\text{ kg}\cdot\text{m}^2/\text{s}}{4.0\text{ kg}\cdot\text{m}^2} = -1.0\text{ rad/s}$$

Evaluating the total kinetic energy of the system after the collision shows that it is less than that before the collision because kinetic energy is not conserved in an inelastic collision.

Table 11.1

Comparison of Values in Example 11.10 Before and After the Collision[a]

	v (m/s)	ω (rad/s)	p (kg·m/s)	L (kg·m²/s)	K_{trans} (J)	K_{rot} (J)
Before						
Disk	3.0	—	6.0	−12	9.0	—
Stick	0	0	0	0	0	0
Total for System	—	—	6.0	−12	9.0	0
After						
Disk	2.3	—	4.7	−9.3	5.4	—
Stick	1.3	−2.0	1.3	−2.7	0.9	2.7
Total for System	—	—	6.0	−12	6.3	2.7

[a] Notice that linear momentum, angular momentum, and total kinetic energy are conserved.

At the Interactive Worked Example link at **http://www.pse6.com,** *you can adjust the speed and position of the disk and observe the collision.*

11.5 The Motion of Gyroscopes and Tops

A very unusual and fascinating type of motion you probably have observed is that of a top spinning about its axis of symmetry, as shown in Figure 11.14a. If the top spins very rapidly, the symmetry axis rotates about the z axis, sweeping out a cone (see Fig. 11.14b). The motion of the symmetry axis about the vertical—known as **precessional motion**—is usually slow relative to the spinning motion of the top.

Precessional motion

It is quite natural to wonder why the top does not fall over. Because the center of mass is not directly above the pivot point O, a net torque is clearly acting on the top about O—a torque resulting from the gravitational force $M\mathbf{g}$. The top would certainly fall over if it were not spinning. Because it is spinning, however, it has an angular momentum $\mathbf{L}$ directed along its symmetry axis. We shall show that this symmetry axis moves about the z axis (precessional motion occurs) because the torque produces a change in the *direction* of the symmetry axis. This is an excellent example of the importance of the directional nature of angular momentum.

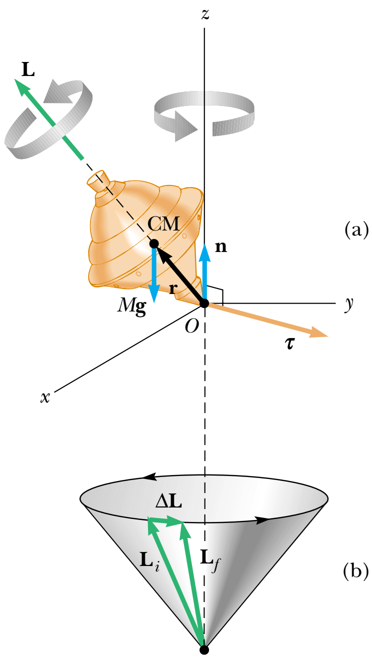


Figure 11.14 Precessional motion of a top spinning about its symmetry axis. (a) The only external forces acting on the top are the normal force $\mathbf{n}$ and the gravitational force $M\mathbf{g}$. The direction of the angular momentum $\mathbf{L}$ is along the axis of symmetry. The right-hand rule indicates that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times M\mathbf{g}$ is in the xy plane. (b). The direction of $\Delta\mathbf{L}$ is parallel to that of $\boldsymbol{\tau}$ in part (a). The fact that $\mathbf{L}_f = \Delta\mathbf{L} + \mathbf{L}_i$ indicates that the top precesses about the z axis.

The essential features of precessional motion can be illustrated by considering the simple gyroscope shown in Figure 11.15a. The two forces acting on the top are the downward gravitational force $M\mathbf{g}$ and the normal force $\mathbf{n}$ acting upward at the pivot point O. The normal force produces no torque about the pivot because its moment arm through that point is zero. However, the gravitational force produces a torque $\boldsymbol{\tau} = \mathbf{r} \times M\mathbf{g}$ about O, where the direction of $\boldsymbol{\tau}$ is perpendicular to the plane formed by $\mathbf{r}$ and $M\mathbf{g}$. By necessity, the vector $\boldsymbol{\tau}$ lies in a horizontal xy plane perpendicular to the angular momentum vector. The net torque and angular momentum of the gyroscope are related through Equation 11.13:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

From this expression, we see that the nonzero torque produces a change in angular momentum $d\mathbf{L}$—a change that is in the same direction as $\boldsymbol{\tau}$. Therefore, like the torque vector, $d\mathbf{L}$ must also be perpendicular to $\mathbf{L}$. Figure 11.15b illustrates the resulting precessional motion of the symmetry axis of the gyroscope. In a time interval dt, the change in angular momentum is $d\mathbf{L} = \mathbf{L}_f - \mathbf{L}_i = \boldsymbol{\tau}\, dt$. Because $d\mathbf{L}$ is perpendicular to $\mathbf{L}$, the magnitude of $\mathbf{L}$ does not change ($|\mathbf{L}_i| = |\mathbf{L}_f|$). Rather, what is changing is the *direction* of $\mathbf{L}$. Because the change in angular momentum $d\mathbf{L}$ is in the direction of $\boldsymbol{\tau}$, which lies in the xy plane, the gyroscope undergoes precessional motion.

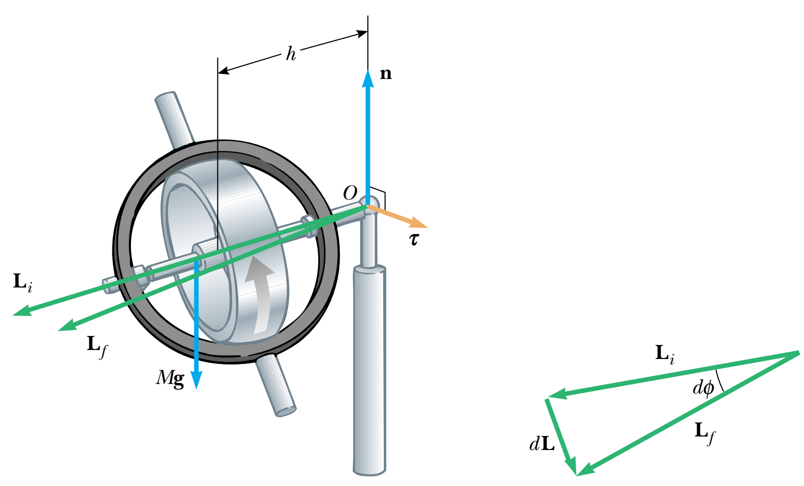


Figure 11.15 (a) The motion of a simple gyroscope pivoted a distance h from its center of mass. The gravitational force $M\mathbf{g}$ produces a torque about the pivot, and this torque is perpendicular to the axle. (b) Overhead view of the initial and final angular momentum vectors. The torque results in a change in angular momentum $d\mathbf{L}$ in a direction perpendicular to the axle. The axle sweeps out an angle $d\phi$ in a time interval dt.

To simplify the description of the system, we must make an assumption: The total angular momentum of the precessing wheel is the sum of the angular momentum $I\boldsymbol{\omega}$ due to the spinning and the angular momentum due to the motion of the center of mass about the pivot. In our treatment, we shall neglect the contribution from the center-of-mass motion and take the total angular momentum to be just $I\boldsymbol{\omega}$. In practice, this is a good approximation if $\boldsymbol{\omega}$ is made very large.

The vector diagram in Figure 11.15b shows that in the time interval dt, the angular momentum vector rotates through an angle $d\phi$, which is also the angle through which the axle rotates. From the vector triangle formed by the vectors $\mathbf{L}_i$, $\mathbf{L}_f$, and $d\mathbf{L}$, we see that

$$\sin(d\phi) \approx d\phi = \frac{dL}{L} = \frac{\tau\, dt}{L} = \frac{(Mgh)\,dt}{L}$$

where we have used the fact that, for small values of any angle θ, $\sin\,\theta \approx \theta$. Dividing through by dt and using the relationship $L = I\omega$, we find that the rate at which the axle rotates about the vertical axis is

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgh}{I\omega} \qquad (11.20)$$

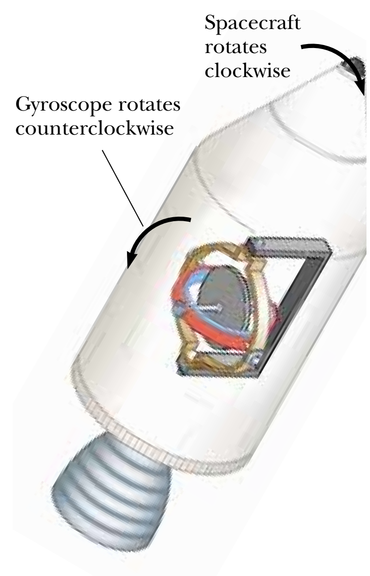
The angular speed ω_p is called the **precessional frequency.** This result is valid only when $\omega_p \ll \omega$. Otherwise, a much more complicated motion is involved. As you can see from Equation 11.20, the condition $\omega_p \ll \omega$ is met when ω is large, that is, the wheel spins rapidly. Furthermore, note that the precessional frequency decreases as ω increases—that is, as the wheel spins faster about its axis of symmetry.

As an example of the usefulness of gyroscopes, suppose you are in a spacecraft in deep space and you need to alter your trajectory. You need to turn the spacecraft around in order to fire the engines in the correct direction. But how do you turn a spacecraft around in empty space? One way is to have small rocket engines that fire perpendicularly out the side of the spacecraft, providing a torque around its center of mass. This is desirable, and many spacecraft have such rockets.

Let us consider another method, however, that is related to angular momentum and does not require the consumption of rocket fuel. Suppose that the spacecraft carries a gyroscope that is not rotating, as in Figure 11.16a. In this case, the angular momentum of the spacecraft about its center of mass is zero. Suppose the gyroscope is set into rotation, giving the gyroscope a nonzero angular momentum. There is no external torque on the isolated system (spacecraft + gyroscope), so the angular momentum of this system must remain zero according to the principle of conservation of angular momentum. This principle can be satisfied if the spacecraft rotates in the direction opposite to that of the gyroscope, so that the angular momentum vectors of the gyroscope and the spacecraft cancel, resulting in no angular momentum of the system. The result of rotating the gyroscope, as in Figure 11.16b, is that the spacecraft turns around! By including three gyroscopes with mutually perpendicular axles, any desired rotation in space can be achieved.

This effect created an undesirable situation with the *Voyager 2* spacecraft during its flight. The spacecraft carried a tape recorder whose reels rotated at high speeds. Each time the tape recorder was turned on, the reels acted as gyroscopes, and the spacecraft started an undesirable rotation in the opposite direction. This had to be counteracted by Mission Control by using the sideward-firing jets to stop the rotation!

(a)



(b)

Figure 11.16 (a) A spacecraft carries a gyroscope that is not spinning. (b) When the gyroscope is set into rotation, the spacecraft turns the other way so that the angular momentum of the system is conserved.

11.6 Angular Momentum as a Fundamental Quantity

We have seen that the concept of angular momentum is very useful for describing the motion of macroscopic systems. However, the concept also is valid on a submicroscopic scale and has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it has been found that the angular momentum of a system is a fundamental quantity. The word *fundamental* in this context implies that angular momentum is an intrinsic property of atoms, molecules, and their constituents, a property that is a part of their very nature.

To explain the results of a variety of experiments on atomic and molecular systems, we rely on the fact that the angular momentum has discrete values. These discrete values are multiples of the fundamental unit of angular momentum $\hbar = h/2\pi$, where h is called Planck's constant:

$$\text{Fundamental unit of angular momentum} = \hbar = 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$$

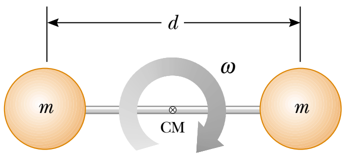


Figure 11.17 The rigid-rotor model of a diatomic molecule. The rotation occurs about the center of mass in the plane of the page.

Let us accept this postulate without proof for the time being and show how it can be used to estimate the angular speed of a diatomic molecule. Consider the O_2 molecule as a rigid rotor, that is, two atoms separated by a fixed distance d and rotating about the center of mass (Fig. 11.17). Equating the angular momentum to the fundamental unit $\hbar$, we can find the order of magnitude of the lowest angular speed:

$$I_{\text{CM}}\omega \approx \hbar \qquad \text{or} \qquad \omega \approx \frac{\hbar}{I_{\text{CM}}}$$

In Example 10.3, we found that the moment of inertia of the O_2 molecule about this axis of rotation is $1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2$. Therefore,

$$\omega \approx \frac{\hbar}{I_{\text{CM}}} = \frac{1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2} \sim 10^{12}\ \text{rad/s}$$

Actual angular speeds are found to be multiples of a number with this order of magnitude.

This simple example shows that certain classical concepts and models, when properly modified, are useful in describing some features of atomic and molecular systems. A wide variety of phenomena on the submicroscopic scale can be explained only if we assume discrete values of the angular momentum associated with a particular type of motion.

The Danish physicist Niels Bohr (1885–1962) accepted and adopted this radical idea of discrete angular momentum values in developing his theory of the hydrogen atom. Strictly classical models were unsuccessful in describing many of the hydrogen atom's properties. Bohr postulated that the electron could occupy only those circular orbits about the proton for which the orbital angular momentum was equal to $n\hbar$, where n is an integer. That is, he made the bold claim that orbital angular momentum is quantized. One can use this simple model to estimate the rotational frequencies of the electron in the various orbits (see Problem 42).

SUMMARY

Take a practice test for this chapter by clicking on the Practice Test link at http://www.pse6.com.

The **torque** $\boldsymbol{\tau}$ due to a force $\mathbf{F}$ about an origin in an inertial frame is defined to be

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \qquad (11.1)$$

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, the **cross product** $\mathbf{A} \times \mathbf{B}$ is a vector $\mathbf{C}$ having a magnitude

$$C \equiv AB \sin\theta \qquad (11.3)$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. The direction of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$, and this direction is determined by the right-hand rule.

The **angular momentum** $\mathbf{L}$ of a particle having linear momentum $\mathbf{p} = m\mathbf{v}$ is

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \qquad (11.10)$$

where $\mathbf{r}$ is the vector position of the particle relative to an origin in an inertial frame.

The **net external torque** acting on a system is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}_{\text{tot}}}{dt} \qquad (11.13)$$

The z component of **angular momentum** of a rigid object rotating about a fixed z axis is

$$L_z = I\omega \qquad (11.14)$$

where I is the moment of inertia of the object about the axis of rotation and ω is its angular speed.

The **net external torque** acting on a rigid object equals the product of its moment of inertia about the axis of rotation and its angular acceleration:

$$\sum \boldsymbol{\tau}_{\text{ext}} = I\alpha \qquad (11.16)$$

If the net external torque acting on a system is zero, then the total angular momentum of the system is constant:

$$\mathbf{L}_i = \mathbf{L}_f \qquad (11.18)$$

Applying this **law of conservation of angular momentum** to a system whose moment of inertia changes gives

$$I_i\omega_i = I_f\omega_f = \text{constant} \qquad (11.19)$$

QUESTIONS

1. Is it possible to calculate the torque acting on a rigid object without specifying an axis of rotation? Is the torque independent of the location of the axis of rotation?

2. Is the triple product defined by $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ a scalar or a vector quantity? Explain why the operation $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ has no meaning.

3. Vector $\mathbf{A}$ is in the negative y direction, and vector $\mathbf{B}$ is in the negative x direction. What are the directions of (a) $\mathbf{A} \times \mathbf{B}$ (b) $\mathbf{B} \times \mathbf{A}$?

4. If a single force acts on an object and the torque caused by the force is nonzero about some point, is there any other point about which the torque is zero?

5. Suppose that the vector velocity of a particle is completely specified. What can you conclude about the direction of its angular momentum vector with respect to the direction of motion?

6. If a system of particles is in motion, is it possible for the total angular momentum to be zero about some origin? Explain.

7. If the torque acting on a particle about a certain origin is zero, what can you say about its angular momentum about that origin?

8. A ball is thrown in such a way that it does not spin about its own axis. Does this mean that the angular momentum is zero about an arbitrary origin? Explain.

9. For a helicopter to be stable as it flies, it must have at least two propellers. Why?

10. A particle is moving in a circle with constant speed. Locate one point about which the particle's angular momentum is constant and another point about which it changes in time.

11. Why does a long pole help a tightrope walker stay balanced?

12. Often when a high diver wants to turn a flip in midair, she draws her legs up against her chest. Why does this make her rotate faster? What should she do when she wants to come out of her flip?

13. In some motorcycle races, the riders drive over small hills, and the motorcycle becomes airborne for a short time. If the motorcycle racer keeps the throttle open while leaving the hill and going into the air, the motorcycle tends to nose upward. Why does this happen?

14. Stars originate as large bodies of slowly rotating gas. Because of gravitation, these clumps of gas slowly decrease in size. What happens to the angular speed of a star as it shrinks? Explain.

15. If global warming occurs over the next century, it is likely that some polar ice will melt and the water will be distributed closer to the Equator. How would this change the moment of inertia of the Earth? Would the length of the day (one revolution) increase or decrease?

16. A mouse is initially at rest on a horizontal turntable mounted on a frictionless vertical axle. If the mouse

Figure Q11.17

begins to walk clockwise around the perimeter, what happens to the turntable? Explain.

17. A cat usually lands on its feet regardless of the position from which it is dropped. A slow-motion film of a cat falling shows that the upper half of its body twists in one direction while the lower half twists in the opposite direction. (See Figure Q11.17.) Why does this type of rotation occur?

18. As the cord holding a tether ball winds around a thin pole, what happens to the angular speed of the ball? Explain.

19. If you toss a textbook into the air, rotating it each time about one of the three axes perpendicular to the textbook, you will find that it will not rotate smoothly about one of these axis. (Try placing a strong rubber band around the book before the toss so it will stay closed.) Its rotation is stable about those axes having the largest and smallest moment of inertia but unstable about the axis of intermediate moment. Try this on your own to find the axis that has this intermediate moment.

20. A scientist arriving at a hotel asks a bellhop to carry a heavy suitcase. When the bellhop rounds a corner, the suitcase suddenly swings away from him for some unknown reason. The alarmed bellhop drops the suitcase and runs away. What might be in the suitcase?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*

= coached solution with hints available at http://www.pse6.com = computer useful in solving problem

= paired numerical and symbolic problems

Section 11.1 The Vector Product and Torque

1. Given $\mathbf{M} = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{N} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, calculate the vector product $\mathbf{M} \times \mathbf{N}$.

2. The vectors 42.0 cm at 15.0° and 23.0 cm at 65.0° both start from the origin. Both angles are measured counterclockwise from the *x* axis. The vectors form two sides of a parallelogram. (a) Find the area of the parallelogram. (b) Find the length of its longer diagonal.

3. Two vectors are given by $\mathbf{A} = -3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ and $\mathbf{B} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between **A** and **B**.

4. Two vectors are given by $\mathbf{A} = -3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{B} = 6\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$. Evaluate the quantities (a) $\cos^{-1}[\mathbf{A}\cdot\mathbf{B}/AB]$ and (b) $\sin^{-1}[|\mathbf{A} \times \mathbf{B}|/AB]$. (c) Which give(s) the angle between the vectors?

5. The wind exerts on a flower the force 0.785 N horizontally to the east. The stem of the flower is 0.450 m long and tilts toward the east, making an angle of 14.0° with the vertical. Find the vector torque of the wind force about the base of the stem.

6. A student claims that she has found a vector **A** such that $(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \times \mathbf{A} = (4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$. Do you believe this claim? Explain.

7. If $|\mathbf{A} \times \mathbf{B}| = \mathbf{A}\cdot\mathbf{B}$, what is the angle between **A** and **B**?

8. A particle is located at the vector position $\mathbf{r} = (\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ m, and the force acting on it is $\mathbf{F} = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$ N. What is the torque about (a) the origin and (b) the point having coordinates (0, 6) m?

9. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act along the two sides of an equilateral triangle as shown in Figure P11.9. Point *O* is the intersection of the altitudes of the triangle. Find a third force $\mathbf{F}_3$ to be applied at *B* and along *BC* that will make the total torque

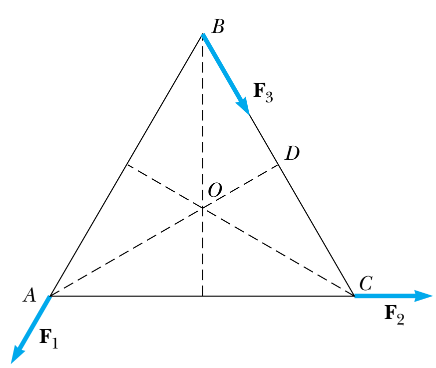


Figure P11.9

zero about the point O. **What If?** Will the total torque change if $\mathbf{F}_3$ is applied not at B but at any other point along BC?

10. Use the definition of the vector product and the definitions of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ to prove Equations 11.7. You may assume that the x axis points to the right, the y axis up, and the z axis toward you (not away from you). This choice is said to make the coordinate system *right-handed.*

Section 11.2 Angular Momentum

11. A light rigid rod 1.00 m in length joins two particles, with masses 4.00 kg and 3.00 kg, at its ends. The combination rotates in the xy plane about a pivot through the center of the rod (Fig. P11.11). Determine the angular momentum of the system about the origin when the speed of each particle is 5.00 m/s.

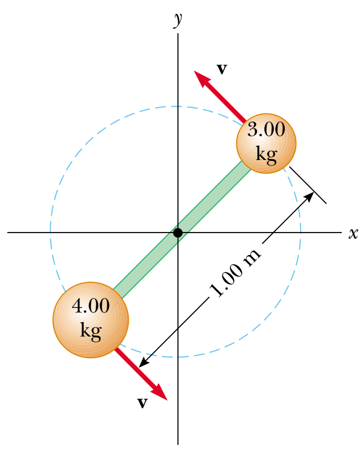


Figure P11.11

12. A 1.50-kg particle moves in the xy plane with a velocity of $\mathbf{v} = (4.20\hat{\mathbf{i}} - 3.60\hat{\mathbf{j}})$ m/s. Determine the angular momentum of the particle when its position vector is $\mathbf{r} = (1.50\hat{\mathbf{i}} + 2.20\hat{\mathbf{j}})$ m.

13. The position vector of a particle of mass 2.00 kg is given as a function of time by $\mathbf{r} = (6.00\hat{\mathbf{i}} + 5.00t\hat{\mathbf{j}})$ m. Determine the angular momentum of the particle about the origin, as a function of time.

14. A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane as shown in Figure P11.14. During the motion, the supporting wire of length ℓ maintains the constant angle θ with the vertical. Show that the magnitude of the angular momentum of the bob about the center of the circle is

$$L = \left(\frac{m^2 g\,\ell^3 \sin^4\theta}{\cos\theta}\right)^{1/2}$$

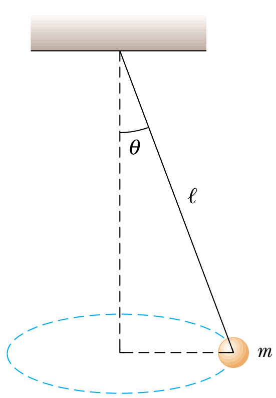


Figure P11.14

15. A particle of mass m moves in a circle of radius R at a constant speed v, as shown in Figure P11.15. If the motion begins at point Q at time $t = 0$, determine the angular momentum of the particle about point P as a function of time.

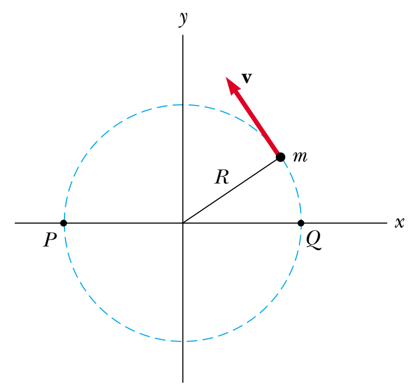


Figure P11.15

16. A 4.00-kg counterweight is attached to a light cord, which is wound around a spool (refer to Fig. 10.20). The spool is a uniform solid cylinder of radius 8.00 cm and mass 2.00 kg. (a) What is the net torque on the system about the point O? (b) When the counterweight has a speed v, the pulley has an angular speed $\omega = v/R$. Determine the total angular momentum of the system about O. (c) Using the fact that $\boldsymbol{\tau} = d\mathbf{L}/dt$ and your result from (b), calculate the acceleration of the counterweight.

17. A particle of mass m is shot with an initial velocity $\mathbf{v}_i$ making an angle θ with the horizontal as shown in Figure P11.17. The particle moves in the gravitational field of the Earth. Find the angular momentum of the particle about the origin when the particle is (a) at the origin, (b) at the highest point of its trajectory, and (c) just before it hits the ground. (d) What torque causes its angular momentum to change?

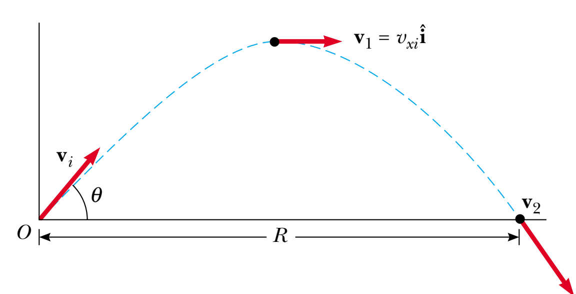


Figure P11.17

18. Heading straight toward the summit of Pike's Peak, an airplane of mass 12 000 kg flies over the plains of Kansas at nearly constant altitude 4.30 km, with a constant velocity of 175 m/s west. (a) What is the airplane's vector angular momentum relative to a wheat farmer on the ground directly below the airplane? (b) Does this value change as the airplane continues its motion along a straight line? (c) **What If?** What is its angular momentum relative to the summit of Pike's Peak?

19. A ball having mass m is fastened at the end of a flagpole that is connected to the side of a tall building at point P shown in Figure P11.19. The length of the flagpole is ℓ and it makes an angle θ with the horizontal. If the ball becomes loose and starts to fall, determine the angular momentum (as a function of time) of the ball about point P. Neglect air resistance.

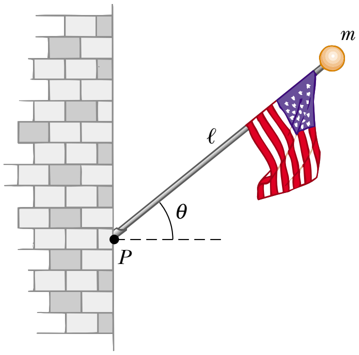


Figure P11.19

20. A fireman clings to a vertical ladder and directs the nozzle of a hose horizontally toward a burning building. The rate of water flow is 6.31 kg/s, and the nozzle speed is 12.5 m/s. The hose passes vertically between the fireman's feet, which are 1.30 m below the nozzle. Choose the origin to be inside the hose between the fireman's feet. What torque must the fireman exert on the hose? That is, what is the rate of change of the angular momentum of the water?

Section 11.3 Angular Momentum of a Rotating Rigid Object

21. Show that the kinetic energy of an object rotating about a fixed axis with angular momentum $L = I\omega$ can be written as $K = L^2/2I$.

22. A uniform solid sphere of radius 0.500 m and mass 15.0 kg turns counterclockwise about a vertical axis through its center. Find its vector angular momentum when its angular speed is 3.00 rad/s.

23. A uniform solid disk of mass 3.00 kg and radius 0.200 m rotates about a fixed axis perpendicular to its face. If the angular frequency of rotation is 6.00 rad/s, calculate the angular momentum of the disk when the axis of rotation (a) passes through its center of mass and (b) passes through a point midway between the center and the rim.

24. Big Ben (Figure P10.40), the Parliament Building tower clock in London, has hour and minute hands with lengths of 2.70 m and 4.50 m and masses of 60.0 kg and 100 kg, respectively. Calculate the total angular momentum of these hands about the center point. Treat the hands as long, thin uniform rods.

25. A particle of mass 0.400 kg is attached to the 100-cm mark of a meter stick of mass 0.100 kg. The meter stick rotates on a horizontal, frictionless table with an angular speed of 4.00 rad/s. Calculate the angular momentum of the system when the stick is pivoted about an axis (a) perpendicular to the table through the 50.0-cm mark and (b) perpendicular to the table through the 0-cm mark.

26. The distance between the centers of the wheels of a motorcycle is 155 cm. The center of mass of the motorcycle, including the biker, is 88.0 cm above the ground and halfway between the wheels. Assume the mass of each wheel is small compared to the body of the motorcycle. The engine drives the rear wheel only. What horizontal acceleration of the motorcycle will make the front wheel rise off the ground?

27. A space station is constructed in the shape of a hollow ring of mass 5.00×10^4 kg. Members of the crew walk on a deck formed by the inner surface of the outer cylindrical wall of the ring, with radius 100 m. At rest when constructed, the ring is set rotating about its axis so that the people inside experience an effective free-fall acceleration equal to g. (Figure P11.27 shows the ring together with some other parts that make a negligible contribution to the total moment of inertia.) The rotation is achieved by firing two small rockets attached tangentially to opposite points on the outside of the ring. (a) What angular momentum does the space station acquire? (b) How long must the rockets be fired if each exerts a thrust of 125 N? (c) Prove that the total torque on the ring, multiplied by the time interval found in part (b), is equal to the change in angular momentum, found in part (a). This equality represents the *angular impulse–angular momentum theorem.*

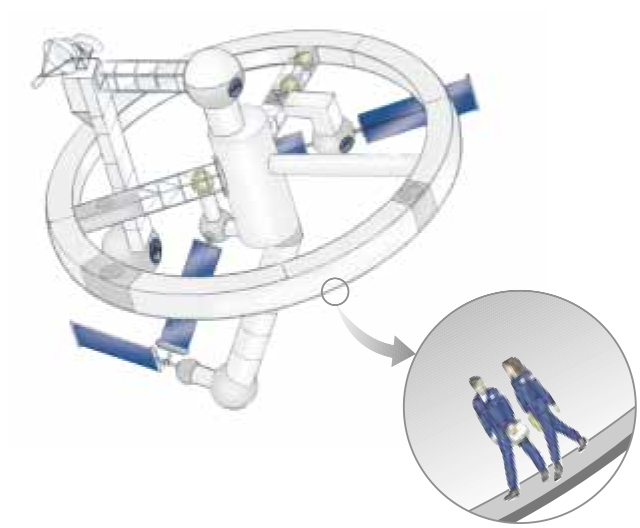


Figure P11.27 Problems 27 and 36.

Section 11.4 Conservation of Angular Momentum

28. A cylinder with moment of inertia I_1 rotates about a vertical, frictionless axle with angular speed ω_i. A second cylinder, this one having moment of inertia I_2 and initially not rotating, drops onto the first cylinder (Fig. P11.28). Because of friction between the surfaces, the two eventually reach the same angular speed ω_f. (a) Calculate ω_f. (b) Show that the kinetic energy of the system decreases in this interaction, and calculate the ratio of the final to the initial rotational energy.

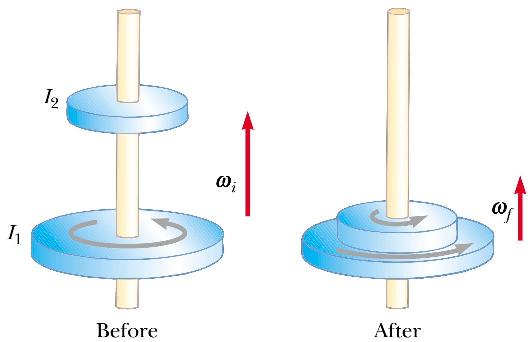


Figure P11.28

29. A playground merry-go-round of radius $R = 2.00$ m has a moment of inertia $I = 250\ \text{kg}\cdot\text{m}^2$ and is rotating at 10.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0-kg child hops onto the merry-go-round and manages to sit down on the edge. What is the new angular speed of the merry-go-round?

30. A student sits on a freely rotating stool holding two weights, each of mass 3.00 kg (Figure P11.30). When his arms are extended horizontally, the weights are 1.00 m from the axis of rotation and he rotates with an angular speed of 0.750 rad/s. The moment of inertia of the student plus stool is $3.00\ \text{kg}\cdot\text{m}^2$ and is assumed to be constant. The student pulls the weights inward horizontally to a position 0.300 m from the rotation axis. (a) Find the new angular speed of the student. (b) Find the kinetic energy of the rotating system before and after he pulls the weights inward.

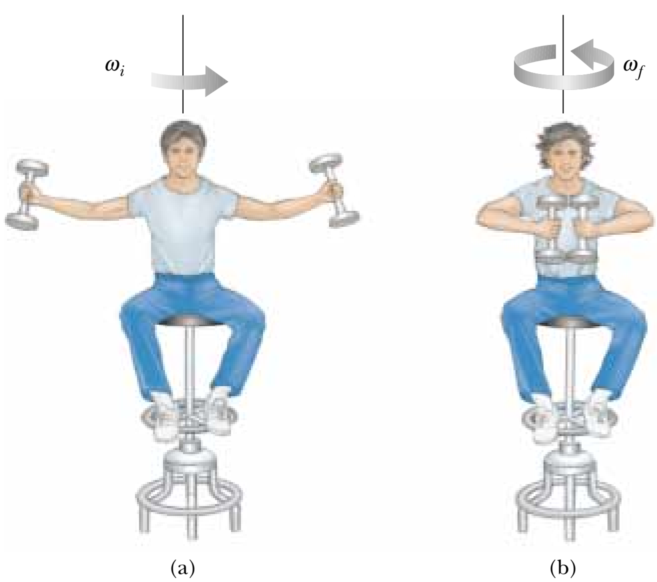


Figure P11.30

31. A uniform rod of mass 100 g and length 50.0 cm rotates in a horizontal plane about a fixed, vertical, frictionless pin through its center. Two small beads, each of mass 30.0 g, are mounted on the rod so that they are able to slide without friction along its length. Initially the beads are held by catches at positions 10.0 cm on each side of center, at which time the system rotates at an angular speed of 20.0 rad/s. Suddenly, the catches are released and the small beads slide outward along the rod. (a) Find the angular speed of the system at the instant the beads reach the ends of the rod. (b) **What if** the beads fly off the ends? What is the angular speed of the rod after this occurs?

32. An umbrella consists of a circle of cloth, a thin rod with the handle at one end and the center of the cloth at the other end, and several straight uniform ribs hinged to the top end of the rod and holding the cloth taut. With the ribs perpendicular to the rod, the umbrella is set rotating about the rod with an angular speed of 1.25 rad/s. The cloth is so light and the rod is so thin that they make negligible contributions to the moment of inertia, in comparison to the ribs. The spinning umbrella is balanced on its handle and keeps rotating without friction. Suddenly its latch breaks and the umbrella partly folds up, until each rib makes an angle of 22.5° with the rod. What is the final angular speed of the umbrella?

33. A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of $500\ \text{kg}\cdot\text{m}^2$ and a radius of 2.00 m. The turntable is initially at rest and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth. (a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

34. A puck of mass 80.0 g and radius 4.00 cm slides along an air table at a speed of 1.50 m/s as shown in Figure P11.34a. It makes a glancing collision with a second puck of radius 6.00 cm and mass 120 g (initially at rest) such that their rims just touch. Because their rims are coated with instant-acting glue, the pucks stick together and spin after the collision (Fig. P11.34b). (a) What is the angular momentum of the system relative to the center of mass? (b) What is the angular speed about the center of mass?

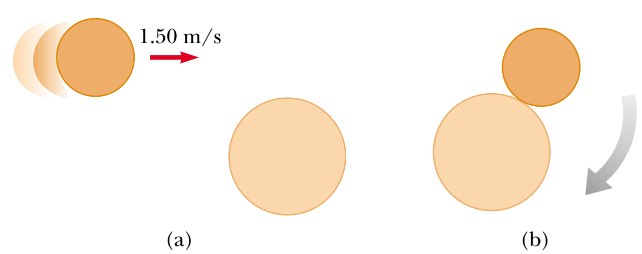


Figure P11.34

35. A wooden block of mass M resting on a frictionless horizontal surface is attached to a rigid rod of length ℓ and of negligible mass (Fig. P11.35). The rod is pivoted at the other end. A bullet of mass m traveling parallel to the horizontal surface and perpendicular to the rod with speed v hits the block and becomes embedded in it. (a) What is the angular momentum of the bullet–block system? (b) What fraction of the original kinetic energy is lost in the collision?

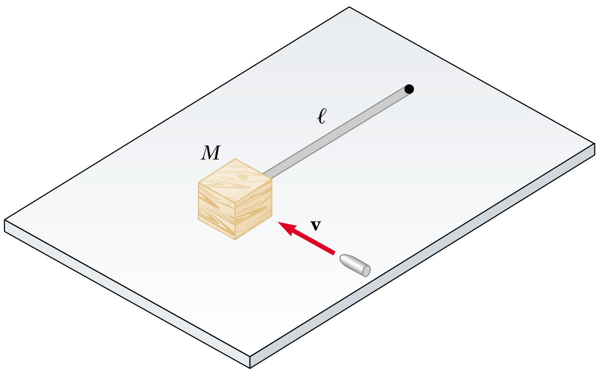


Figure P11.35

36. A space station shaped like a giant wheel has a radius of 100 m and a moment of inertia of 5.00×10^8 kg·m^2. A crew of 150 is living on the rim, and the station's rotation causes the crew to experience an apparent free-fall acceleration of g (Fig. P11.27). When 100 people move to the center of the station for a union meeting, the angular speed changes. What apparent free-fall acceleration is experienced by the managers remaining at the rim? Assume that the average mass for each inhabitant is 65.0 kg.

37. A wad of sticky clay with mass m and velocity $\mathbf{v}_i$ is fired at a solid cylinder of mass M and radius R (Figure P11.37). The cylinder is initially at rest and is mounted on a fixed horizontal axle that runs through its center of mass. The line of motion of the projectile is perpendicular to the axle and at a distance $d < R$ from the center. (a) Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. (b) Is mechanical energy of the clay–cylinder system conserved in this process? Explain your answer.

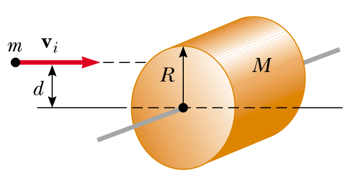


Figure P11.37

38. A thin uniform rectangular sign hangs vertically above the door of a shop. The sign is hinged to a stationary horizontal rod along its top edge. The mass of the sign is 2.40 kg and its vertical dimension is 50.0 cm. The sign is swinging without friction, becoming a tempting target for children armed with snowballs. The maximum angular displacement of the sign is 25.0° on both sides of the vertical. At a moment when the sign is vertical and moving to the left, a snowball of mass 400 g, traveling horizontally with a velocity of 160 cm/s to the right, strikes perpendicularly the lower edge of the sign and sticks there. (a) Calculate the angular speed of the sign immediately before the impact. (b) Calculate its angular speed immediately after the impact. (c) The spattered sign will swing up through what maximum angle?

39. Suppose a meteor of mass 3.00×10^{13} kg, moving at 30.0 km/s relative to the center of the Earth, strikes the Earth. What is the order of magnitude of the maximum possible decrease in the angular speed of the Earth due to this collision? Explain your answer.

Section 11.5 The Motion of Gyroscopes and Tops

40. A spacecraft is in empty space. It carries on board a gyroscope with a moment of inertia of $I_g = 20.0$ kg·m^2 about the axis of the gyroscope. The moment of inertia of the spacecraft around the same axis is $I_s = 5.00 \times 10^5$ kg·m^2. Neither the spacecraft nor the gyroscope is originally rotating. The gyroscope can be powered up in a negligible period of time to an angular speed of 100 s^{-1}. If the orientation of the spacecraft is to be changed by 30.0°, for how long should the gyroscope be operated?

41. The angular momentum vector of a precessing gyroscope sweeps out a cone, as in Figure 11.14b. Its angular speed, called its precessional frequency, is given by $\omega_p = \tau/L$, where τ is the magnitude of the torque on the gyroscope and L is the magnitude of its angular momentum. In the motion called *precession of the equinoxes*, the Earth's axis of rotation precesses about the perpendicular to its orbital plane with a period of 2.58×10^4 yr. Model the Earth as a uniform sphere and calculate the torque on the Earth that is causing this precession.

Section 11.6 Angular Momentum as a Fundamental Quantity

42. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular frequency of the electron's motion.

Additional Problems

43. We have all complained that there aren't enough hours in a day. In an attempt to change that, suppose that all the people in the world line up at the equator, and all start running east at 2.50 m/s relative to the surface of the Earth. By how much does the length of a day increase? Assume that the world population is 5.50×10^9 people with an average mass of 70.0 kg each, and that the Earth is a solid homogeneous sphere. In addition, you may use the approximation $1/(1 - x) \approx 1 + x$ for small x.

44. A skateboarder with his board can be modeled as a particle of mass 76.0 kg, located at his center of mass. As shown in Figure P8.67 on page 248, the skateboarder starts from rest in a crouching position at one lip of a half-pipe (point Ⓐ). The half-pipe forms one half of a cylinder of radius 6.80 m

with its axis horizontal. On his descent, the skateboarder moves without friction and maintains his crouch, so that his center of mass moves through one quarter of a circle of radius 6.30 m. (a) Find his speed at the bottom of the half-pipe (point Ⓑ). (b) Find his angular momentum about the center of curvature. (c) Immediately after passing point Ⓑ, he stands up and raises his arms, lifting his center of gravity from 0.500 m to 0.950 m above the concrete (point Ⓒ). Explain why his angular momentum is constant in this maneuver, while his linear momentum and his mechanical energy are not constant. (d) Find his speed immediately after he stands up, when his center of mass is moving in a quarter circle of radius 5.85 m. (e) What work did the skateboarder's legs do on his body as he stood up? Next, the skateboarder glides upward with his center of mass moving in a quarter circle of radius 5.85 m. His body is horizontal when he passes point Ⓓ, the far lip of the half-pipe. (f) Find his speed at this location. At last he goes ballistic, twisting around while his center of mass moves vertically. (g) How high above point Ⓓ does he rise? (h) Over what time interval is he airborne before he touches down, facing downward and again in a crouch, 2.34 m below the level of point Ⓓ? (i) Compare the solution to this problem with the solution to Problem 8.67. Which is more accurate? Why? (*Caution*: Do not try this yourself without the required skill and protective equipment, or in a drainage channel to which you do not have legal access.)

45. A rigid, massless rod has three particles with equal masses attached to it as shown in Figure P11.45. The rod is free to rotate in a vertical plane about a frictionless axle perpendicular to the rod through the point P, and is released from rest in the horizontal position at $t = 0$. Assuming m and d are known, find (a) the moment of inertia of the system (rod plus particles) about the pivot, (b) the torque acting on the system at $t = 0$, (c) the angular acceleration of the system at $t = 0$, (d) the linear acceleration of the particle labeled 3 at $t = 0$, (e) the maximum kinetic energy of the system, (f) the maximum angular speed reached by the rod, (g) the maximum angular momentum of the system, and (h) the maximum speed reached by the particle labeled 2.

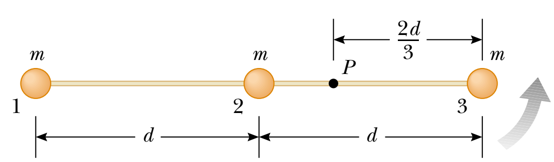


Figure P11.45

46. A 100-kg uniform horizontal disk of radius 5.50 m turns without friction at 2.50 rev/s on a vertical axis through its center, as in Figure P11.46. A feedback mechanism senses the angular speed of the disk, and a drive motor at A maintains the angular speed constant while a 1.20 kg block on top of the disk slides outward in a radial slot. The 1.20-kg block starts at the center of the disk at time $t = 0$ and moves outward with constant speed 1.25 cm/s relative to the disk until it reaches the edge at $t = 440$ s. The sliding block feels no friction. Its motion is constrained to have constant radial speed by a brake at B, producing tension in a light string tied to the block. (a) Find the torque that the drive motor must provide as a function of time, while the block is sliding. (b) Find the value of this torque at $t = 440$ s, just before the sliding block finishes its motion. (c) Find the power that the drive motor must deliver as a function of time. (d) Find the value of the power when the sliding block is just reaching the end of the slot. (e) Find the string tension as a function of time. (f) Find the work done by the drive motor during the 440-s motion. (g) Find the work done by the string brake on the sliding block. (h) Find the total work on the system consisting of the disk and the sliding block.

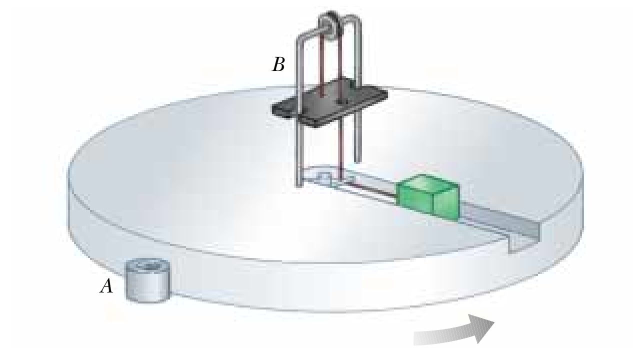


Figure P11.46

47. Comet Halley moves about the Sun in an elliptical orbit, with its closest approach to the Sun being about 0.590 AU and its greatest distance 35.0 AU (1 AU = the Earth–Sun distance). If the comet's speed at closest approach is 54.0 km/s, what is its speed when it is farthest from the Sun? The angular momentum of the comet about the Sun is conserved, because no torque acts on the comet. The gravitational force exerted by the Sun has zero moment arm.

48. A light rope passes over a light, frictionless pulley. One end is fastened to a bunch of bananas of mass M, and a monkey of mass M clings to the other end (Fig. P11.48). The mon-

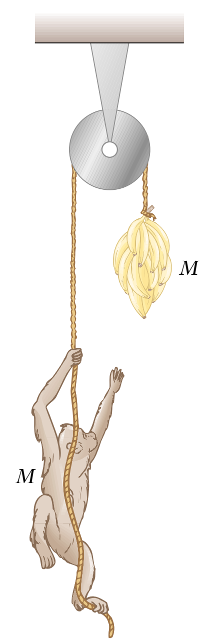


Figure P11.48

key climbs the rope in an attempt to reach the bananas. (a) Treating the system as consisting of the monkey, bananas, rope, and pulley, evaluate the net torque about the pulley axis. (b) Using the results of (a), determine the total angular momentum about the pulley axis and describe the motion of the system. Will the monkey reach the bananas?

49. A puck of mass m is attached to a cord passing through a small hole in a frictionless, horizontal surface (Fig. P11.49). The puck is initially orbiting with speed v_i in a circle of radius r_i. The cord is then slowly pulled from below, decreasing the radius of the circle to r. (a) What is the speed of the puck when the radius is r? (b) Find the tension in the cord as a function of r. (c) How much work W is done in moving m from r_i to r? (*Note:* The tension depends on r.) (d) Obtain numerical values for v, T, and W when $r = 0.100$ m, $m = 50.0$ g, $r_i = 0.300$ m, and $v_i = 1.50$ m/s.

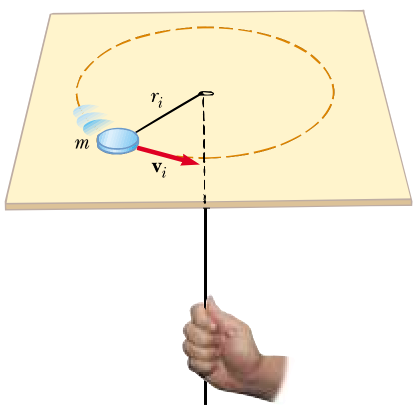


Figure P11.49

50. A projectile of mass m moves to the right with a speed v_i (Fig. P11.50a). The projectile strikes and sticks to the end of a stationary rod of mass M, length d, pivoted about a frictionless axle through its center (Fig. P11.50b). (a) Find the angular speed of the system right after the collision. (b) Determine the fractional loss in mechanical energy due to the collision.

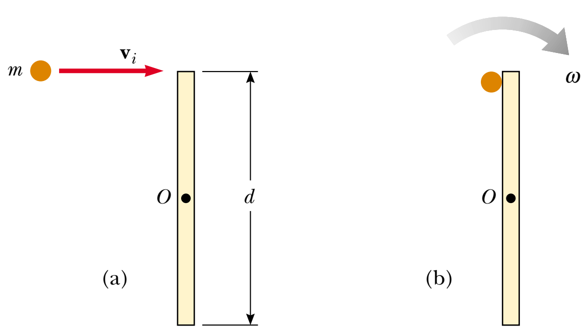


Figure P11.50

51. Two astronauts (Fig. P11.51), each having a mass of 75.0 kg, are connected by a 10.0-m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of 5.00 m/s. Treating the astronauts as particles, calculate (a) the magnitude of the angular momentum of the system and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to 5.00 m. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work does the astronaut do in shortening the rope?

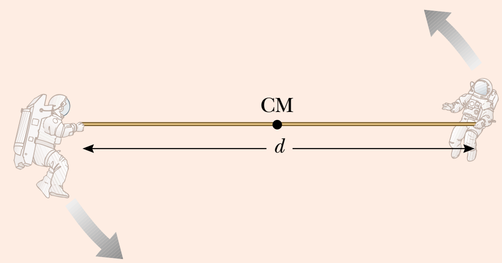


Figure P11.51 Problems 51 and 52.

52. Two astronauts (Fig. P11.51), each having a mass M, are connected by a rope of length d having negligible mass. They are isolated in space, orbiting their center of mass at speeds v. Treating the astronauts as particles, calculate (a) the magnitude of the angular momentum of the system and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to $d/2$. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work does the astronaut do in shortening the rope?

53. Global warming is a cause for concern because even small changes in the Earth's temperature can have significant consequences. For example, if the Earth's polar ice caps were to melt entirely, the resulting additional water in the oceans would flood many coastal cities. Would it appreciably change the length of a day? Calculate the resulting change in the duration of one day. Model the polar ice as having mass 2.30×10^{19} kg and forming two flat disks of radius 6.00×10^5 m. Assume the water spreads into an unbroken thin spherical shell after it melts.

54. A solid cube of wood of side $2a$ and mass M is resting on a horizontal surface. The cube is constrained to rotate about an axis AB (Fig. P11.54). A bullet of mass m and speed v is shot at the face opposite $ABCD$ at a height of $4a/3$. The bullet becomes embedded in the cube. Find the minimum value of v required to tip the cube so that it falls on face $ABCD$. Assume $m \ll M$.

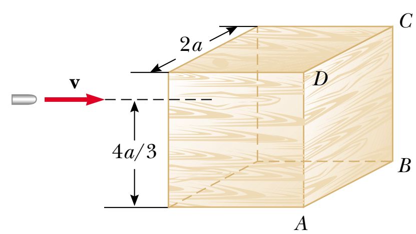


Figure P11.54

55. A solid cube of side $2a$ and mass M is sliding on a frictionless surface with uniform velocity $\mathbf{v}$ as in Figure P11.55a. It hits a small obstacle at the end of the table, which causes the cube to tilt as in Figure P11.55b. Find the minimum value of $\mathbf{v}$ such that the cube falls off the table. Note that the moment of inertia of the cube about an axis along one of its edges is $8Ma^2/3$. (*Hint:* The cube undergoes an inelastic collision at the edge.)

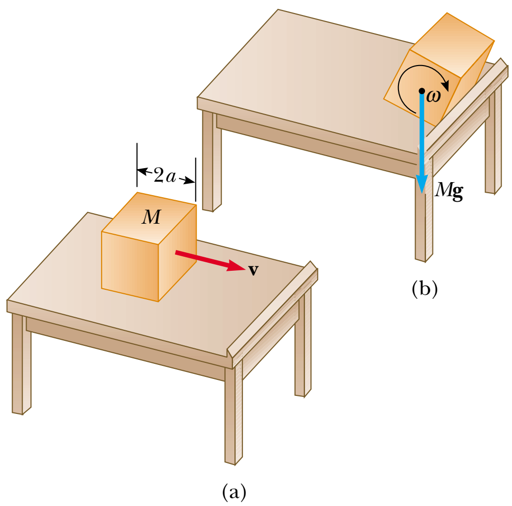


Figure P11.55

56. A uniform solid disk is set into rotation with an angular speed ω_i about an axis through its center. While still rotating at this speed, the disk is placed into contact with a horizontal surface and released as in Figure P11.56. (a) What is the angular speed of the disk once pure rolling takes place? (b) Find the fractional loss in kinetic energy from the time the disk is released until pure rolling occurs. (*Hint:* Consider torques about the center of mass.)

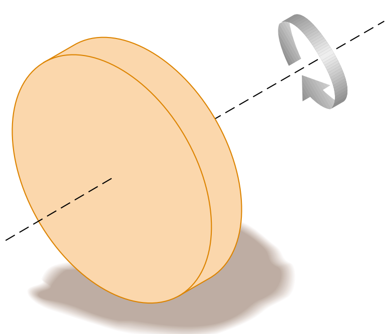


Figure P11.56 Problems 56 and 57.

57. Suppose a solid disk of radius R is given an angular speed ω_i about an axis through its center and then lowered to a horizontal surface and released, as in Problem 56 (Fig. P11.56). Furthermore, assume that the coefficient of friction between disk and surface is μ. (a) Show that the time interval before pure rolling motion occurs is $R\omega_i/3\mu g$. (b) Show that the distance the disk travels before pure rolling occurs is $R^2\omega_i{}^2/18\mu g$.

Answers to Quick Quizzes

11.1 (d). This result can be obtained by replacing $\mathbf{B} \times \mathbf{A}$ with $-(\mathbf{A} \times \mathbf{B})$, according to Equation 11.4.

11.2 (d). Because of the $\sin\theta$ function, $|\mathbf{A} \times \mathbf{B}|$ is either equal to or smaller than AB, depending on the angle θ.

11.3 (a). If $\mathbf{p}$ and $\mathbf{r}$ are parallel or antiparallel, the angular momentum is zero. For a nonzero angular momentum, the linear momentum vector must be offset from the rotation axis.

11.4 (c). The angular momentum is the product of the linear momentum and the perpendicular distance from the rotation axis to the line along which the linear momentum vector lies.

11.5 (b). The hollow sphere has a larger moment of inertia than the solid sphere.

11.6 (a). The diver is an isolated system, so the product $I\omega$ remains constant. Because her moment of inertia decreases, her angular speed increases.

11.7 (a). As the moment of inertia of the diver decreases, the angular speed increases by the same factor. For example, if I goes down by a factor of 2, ω goes up by a factor of 2. The rotational kinetic energy varies as the square of ω. If I is halved, ω^2 increases by a factor of 4 and the energy increases by a factor of 2.